Fluctuations and scaling in turbulent transport and mixing

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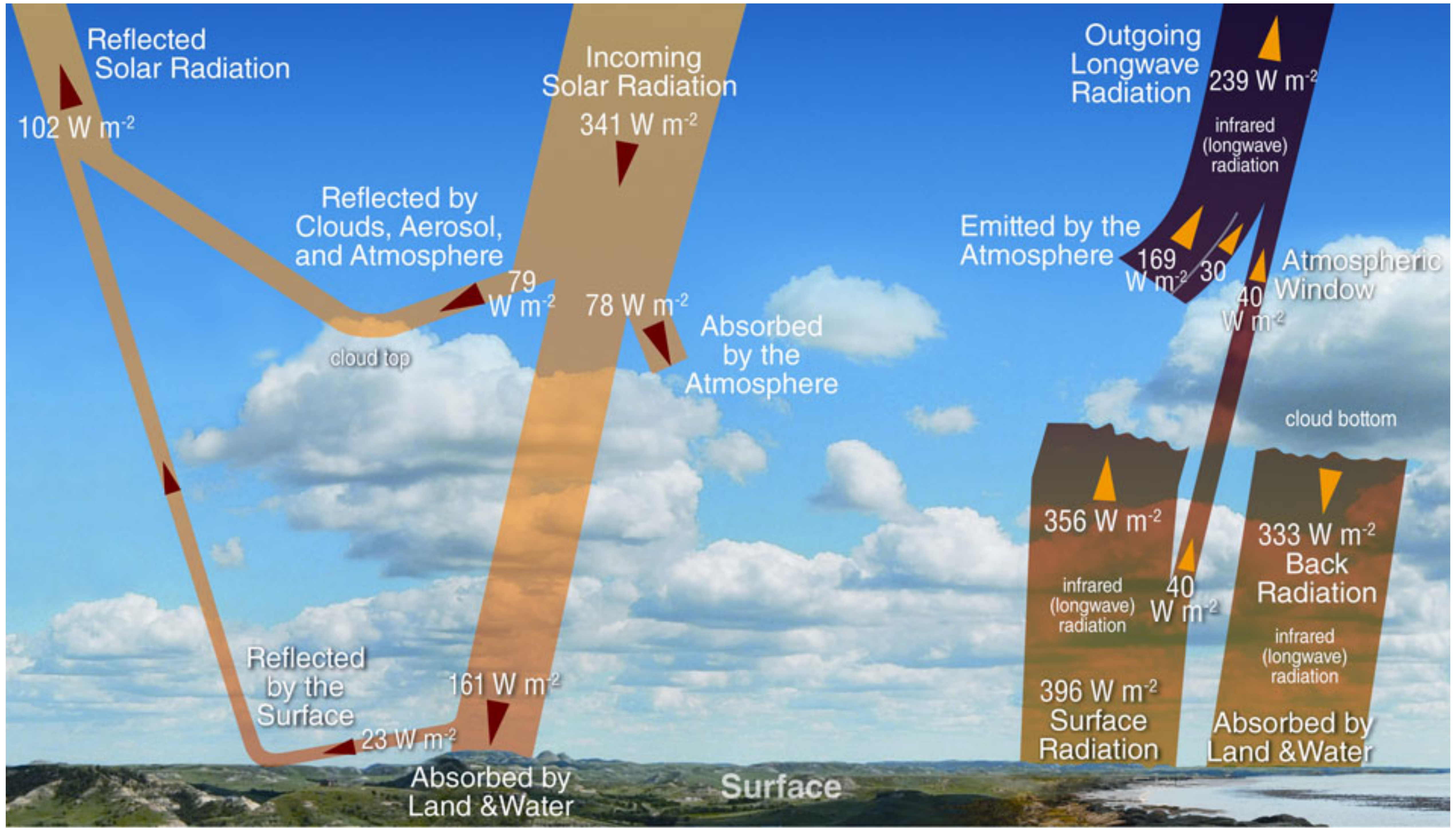
Influence of aerosols on climate

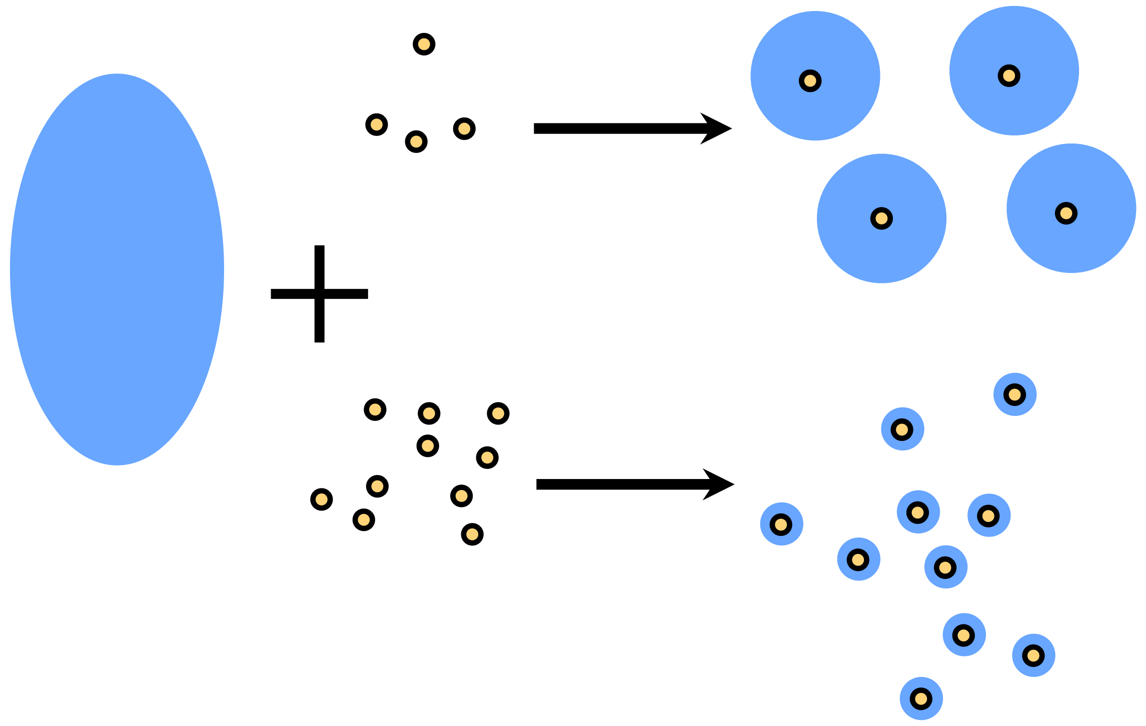
Direct effects

Albedo, Greenhouse

- Lifetime?
- Spatial distribution?
- Scattering properties?

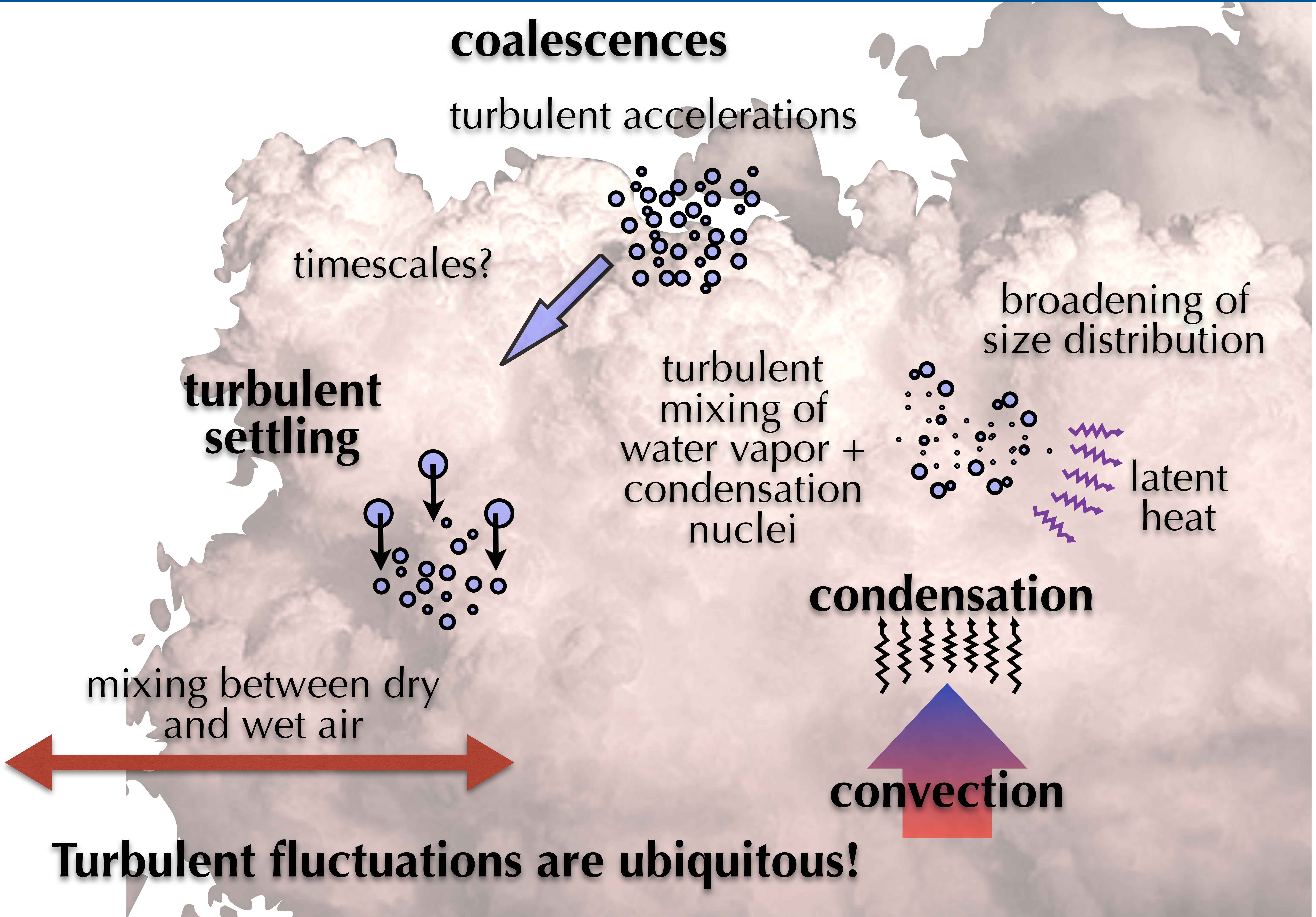


Indirect effects related to their role as condensation nuclei in clouds



- Influence on cloud droplet size distributions?
- Repercussions on the lifecycle of clouds?
- Consequences on global circulation?

Multi-physics of warm clouds



coalescences

turbulent accelerations

timescales?

turbulent settling

broadening of size distribution

turbulent mixing of water vapor + condensation nuclei

latent heat

condensation

mixing between dry and wet air

convection

Turbulent fluctuations are ubiquitous!

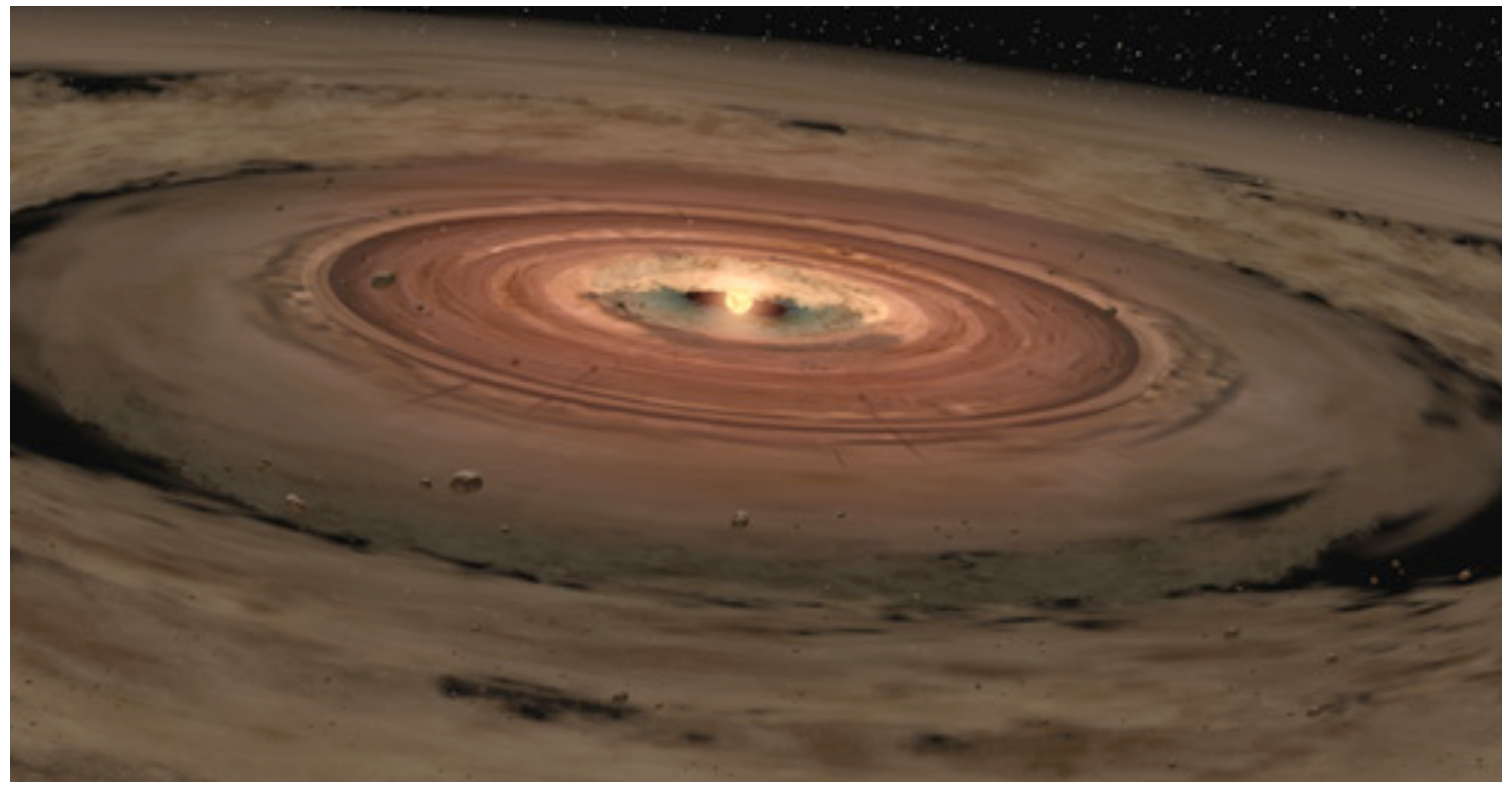
Planet formation

protostar nebula

gravitational collapse

migration toward the equatorial plane

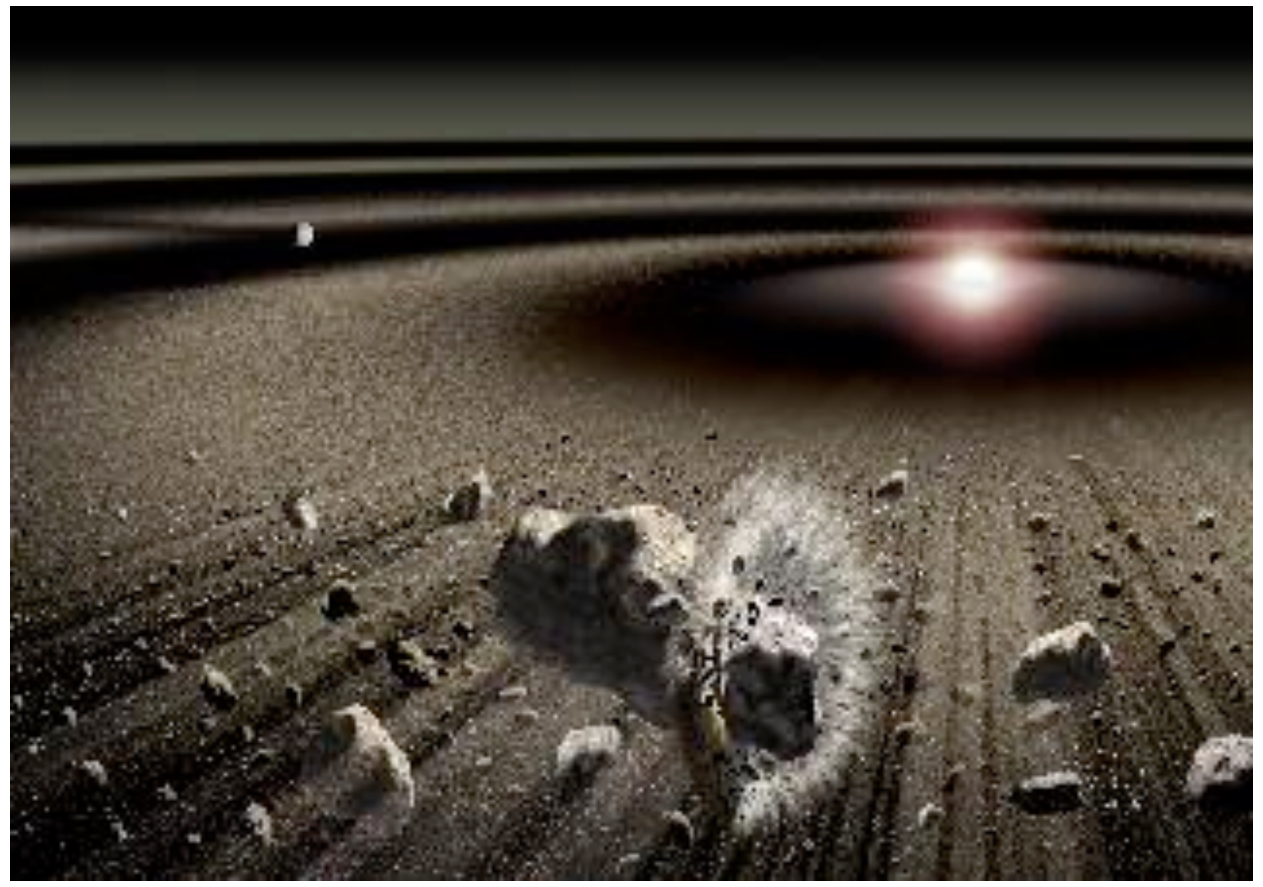
circumstellar disk



Development of **turbulence** in the gas motion + **accretion** of dust particles

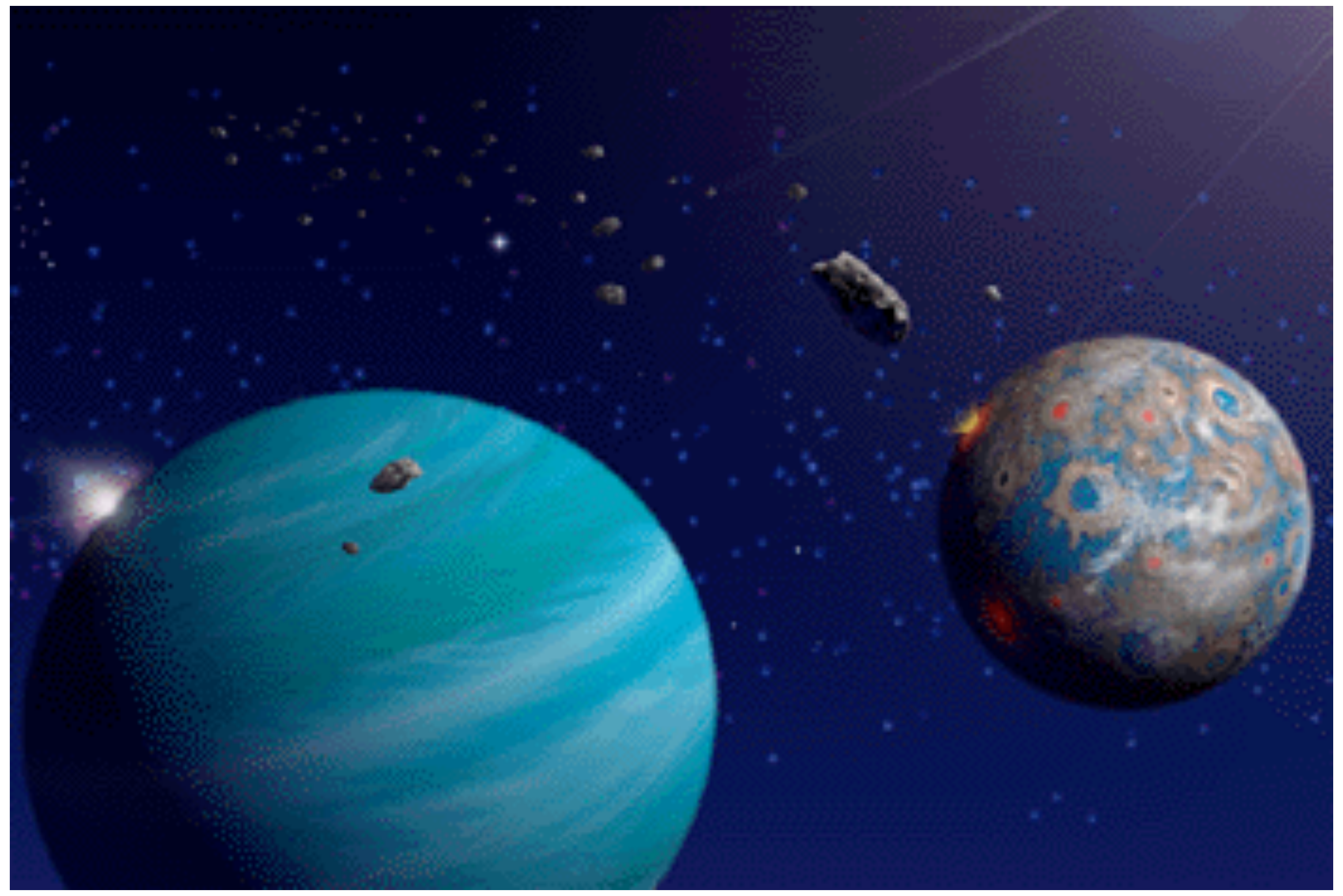
creation of medium-size bodies (mm to m)

Time scales?



gravitational interactions + collisions between large bodies (1m to moons)

planetary system



Atmospheric dispersion

- Fluctuations are important for risk assessments
- **Models/Observations:** space and/or time averages

Content

- **Lecture 1: Richardson 2/3 scaling**
  – Turbulent transport and concentration fluctuations
  – Relation with Lagrangian relative motion
  – Spontaneous stochasticity and dissipative anomaly
  – Richardson law / scaling
  – Models for relative dispersion

- **Lecture 2: Anomalous scaling laws**
  – Intermittency and fronts
  – Kraichnan model and zero modes
  – Coalescences of droplets
  – Breakdown of kinetic models

Length and time scales of turbulence

- Incompressible Navier–Stokes equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho_f}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f},\ \ \nabla\cdot\boldsymbol{u} = 0$$

transfer between scales

dissipation

injection

incompressibility

Reynolds number:

$$Re = \frac{u\,\ell}{\nu} \gg 1$$

measures how weak is viscous dissipation

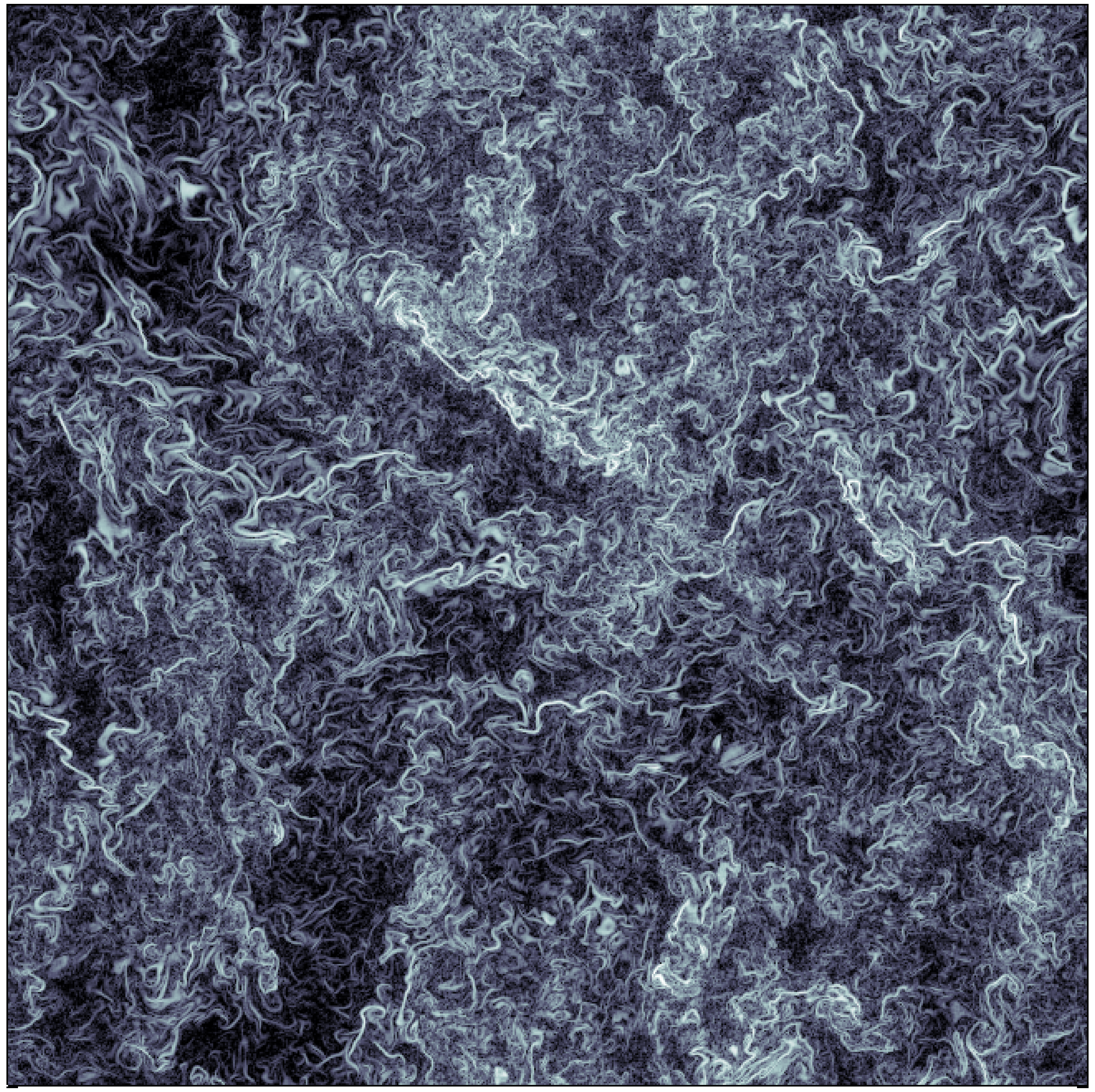
$$Re = (L/\eta)^{4/3}$$

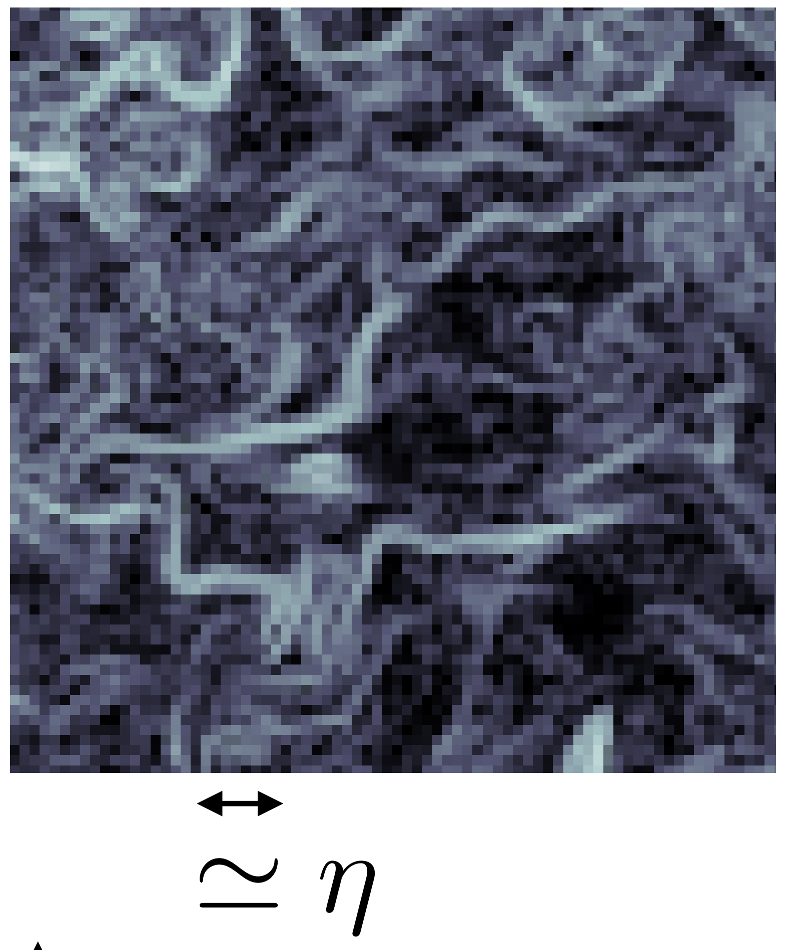
L scale of injection

η dissipative scale (Kolmogorov)

“inertial range”

$$\eta \ll \ell \ll L$$





$\simeq L$

Energy cascades downscale with a ≈ constant rate ε

Kolmogorov 1941 scaling

$$\delta_\ell u = |u(x+\ell) - u(x)| \sim (\varepsilon\ell)^{1/3}$$

$$\tau_\ell = \ell/\delta_\ell u \sim \varepsilon^{-1/3}\ell^{2/3}$$

Advection-diffusion equation

- Concentration field: passive scalar

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + \Phi$$

advection by a prescribed velocity field

diffusion

source

- **Batchelor scale:** $\ell_{\mathrm{B}} = \eta \sqrt{\kappa / \nu}$

$\eta = \varepsilon^{-1/4} \nu^{3/4}$ Kolmogorov viscous dissipative scale

ν fluid kinematic viscosity

ε kinetic energy dissipation rate

ozone in air $\kappa \approx 0.14\,\mathrm{cm}^2\,\mathrm{s}^{-1} \Rightarrow \ell_{\mathrm{B}} \approx 0.8\,\eta \approx 0.8\,\mathrm{mm}$

1µm aerosol $\kappa \approx 2\,.\,10^{-7}\mathrm{cm}^2\,\mathrm{s}^{-1} \Rightarrow \ell_{\mathrm{B}} \approx 10^{-3}\,\eta \approx 1\,\mu\mathrm{m}$

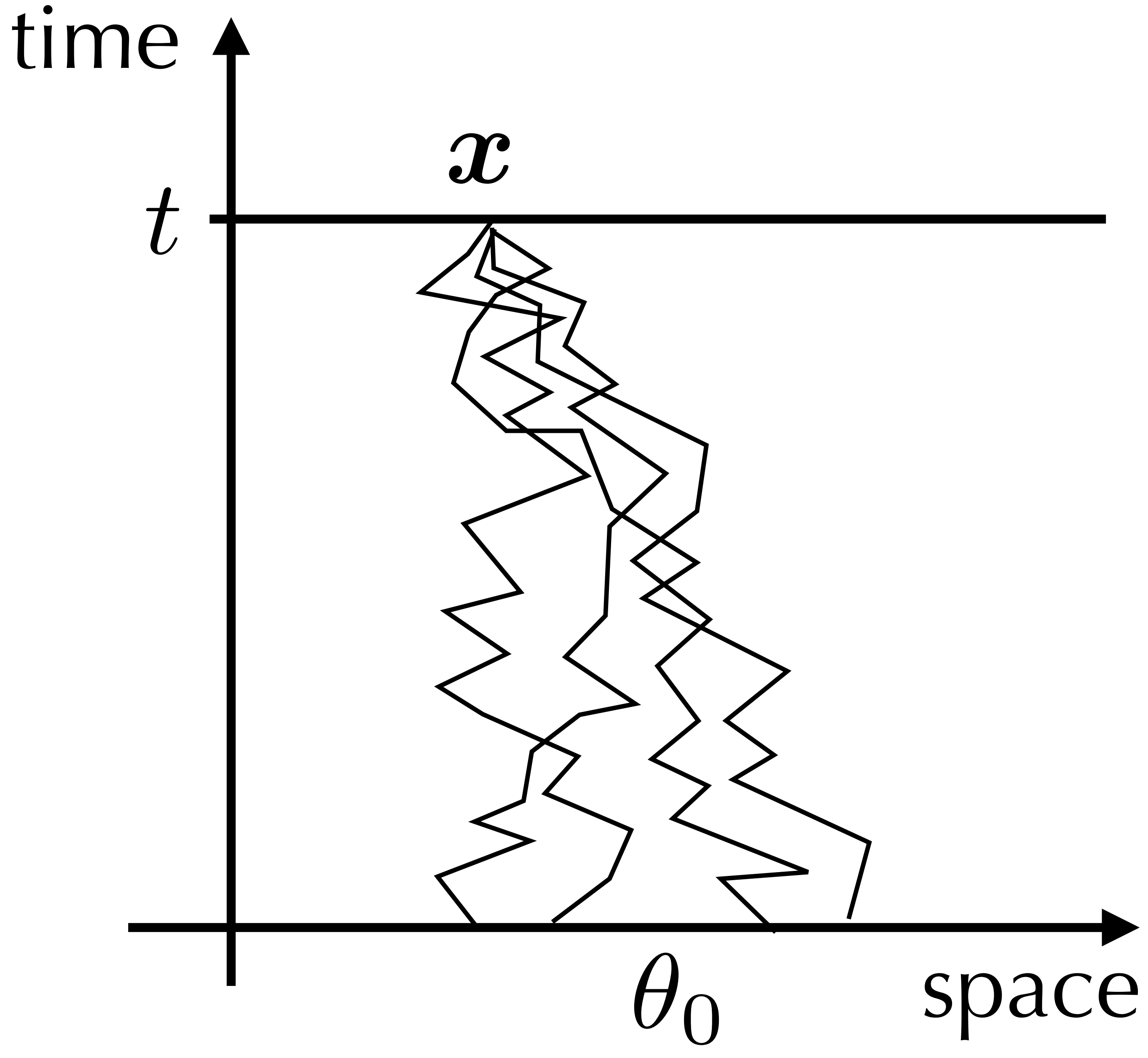
Taylor diffusion

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \qquad \theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x})$$

- Tracers = characteristics of the advection equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t)$$

$$\Rightarrow \; \theta(\boldsymbol{x}, t) = \langle \theta_0(\boldsymbol{x}(0)) \mid \boldsymbol{u} \rangle_\kappa$$



- **Turbulent diffusion** (Taylor 1921)

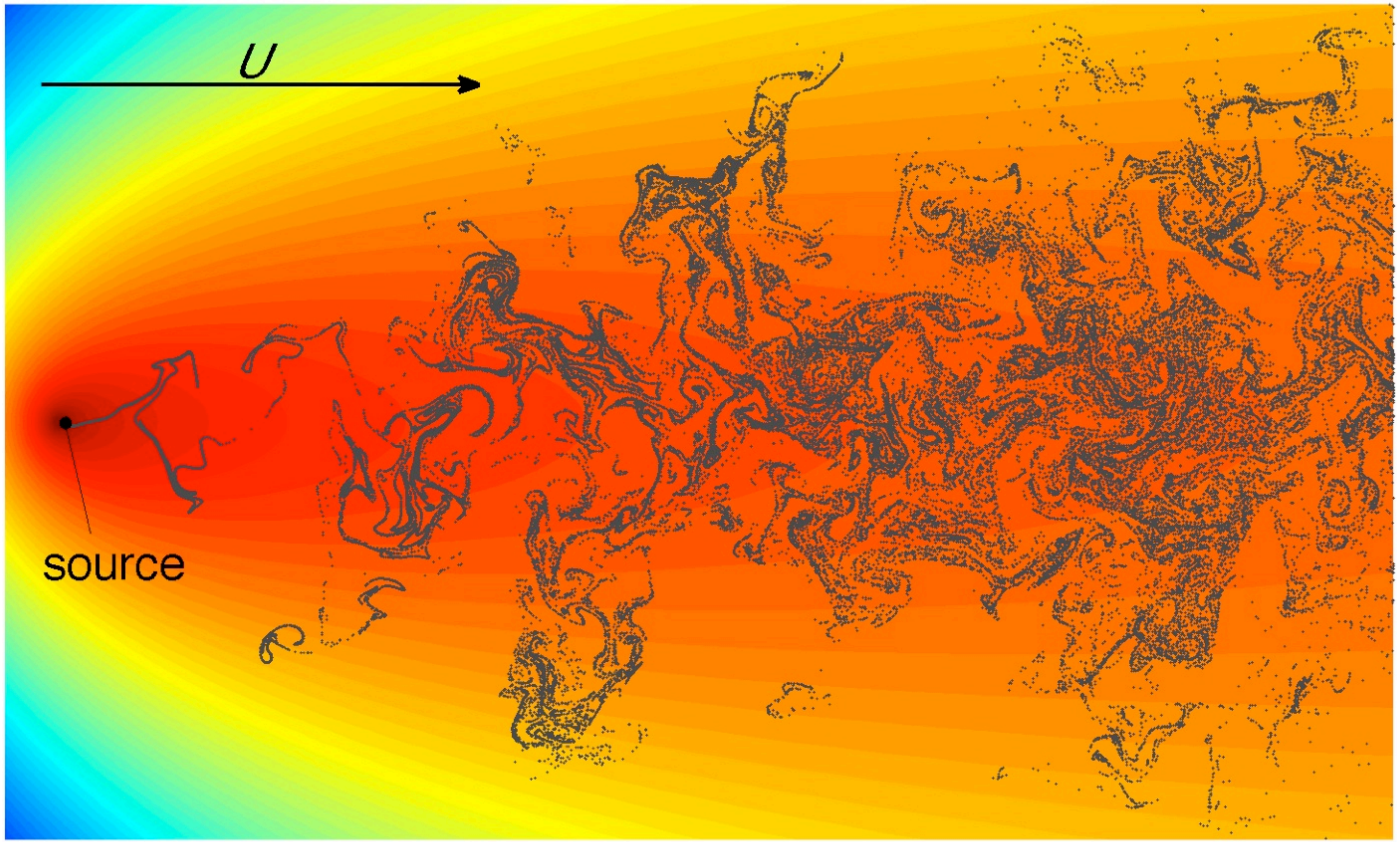
Mean-field description for the averaged concentration

$$\langle |\boldsymbol{x}(t) - \boldsymbol{x}(0)|^2 \rangle = \int_0^t \int_0^t \langle \boldsymbol{u}(\boldsymbol{x}(s), s) \cdot \boldsymbol{u}(\boldsymbol{x}(s'), s') \rangle \,\mathrm{d}s\,\mathrm{d}s' + 2\kappa\, t \simeq 2(T_\mathrm{L} u_\mathrm{rms}^2 + \kappa)\, t$$

$$\Rightarrow \; \partial_t \langle \theta \rangle = -\nabla \cdot \langle \boldsymbol{u}\, \theta \rangle + \kappa \nabla^2 \langle \theta \rangle \approx (\kappa_\mathrm{eff} + \kappa)\, \nabla^2 \langle \theta \rangle$$

Mean vs. meandering plumes

- Averaged concentration is well described by eddy diffusivity



- PDFs have tails rather far from Gaussian
- Spatial correlations relates to relative motion of tracers

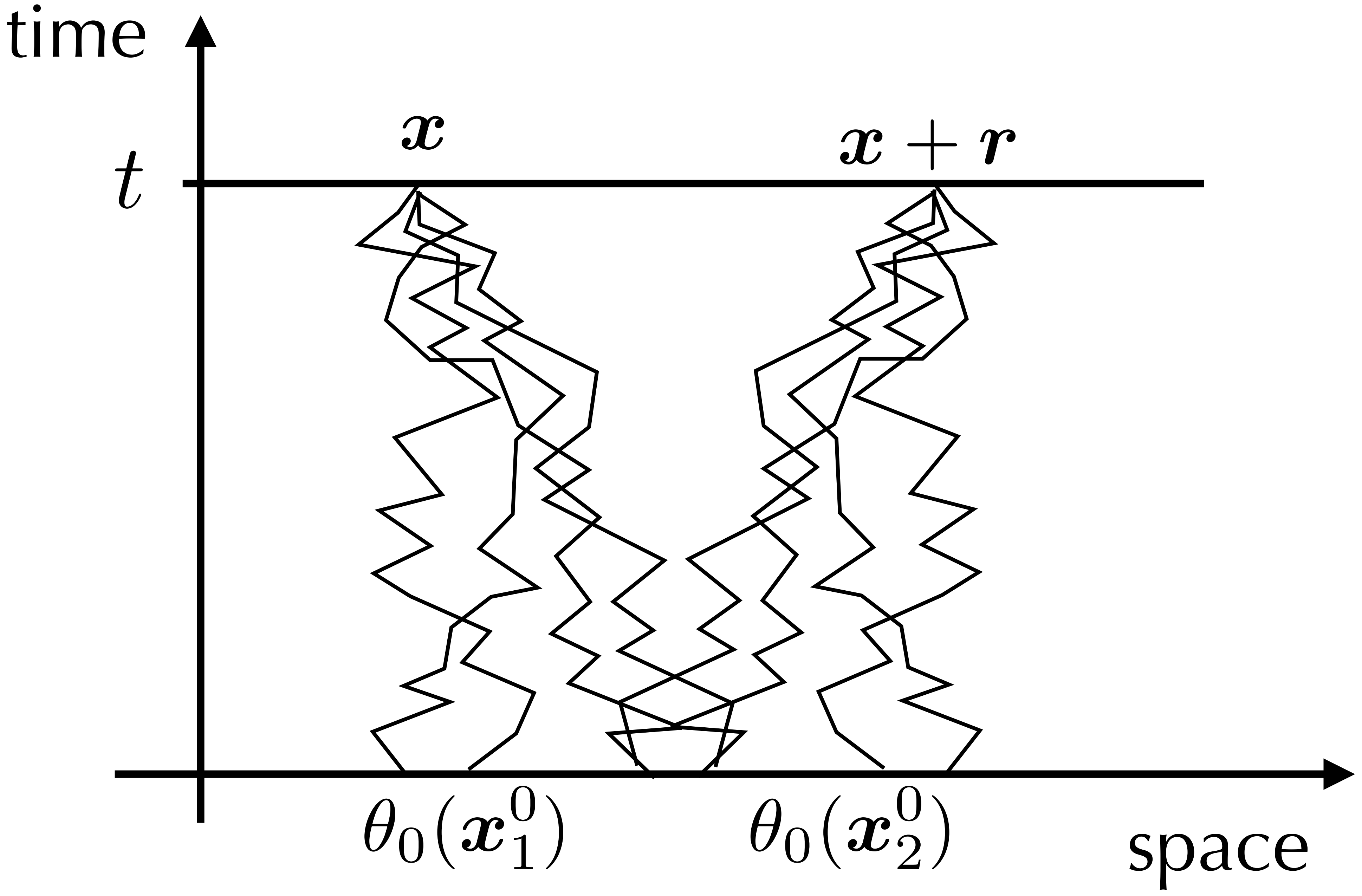
Fluctuations and relative dispersion

- Spatial correlations of the concentration

$$\langle\theta(\boldsymbol{x}+\boldsymbol{r},t)\,\theta(\boldsymbol{x},t)\rangle = \iint \langle\theta_0(\boldsymbol{x}_1^0)\,\theta_0(\boldsymbol{x}_2^0)\rangle\, p_2(\boldsymbol{x}+\boldsymbol{r},\boldsymbol{x},t\,|\,\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)\,\mathrm{d}\boldsymbol{x}_1^0\mathrm{d}\boldsymbol{x}_2^0$$

$p_2(\boldsymbol{x}_1,\boldsymbol{x}_2,t\,|\,\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)$ = joint transition probability density of two tracers $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$

Sawford, *Ann. Rev. Fluid Mech.* 2001



Spontaneous stochasticity

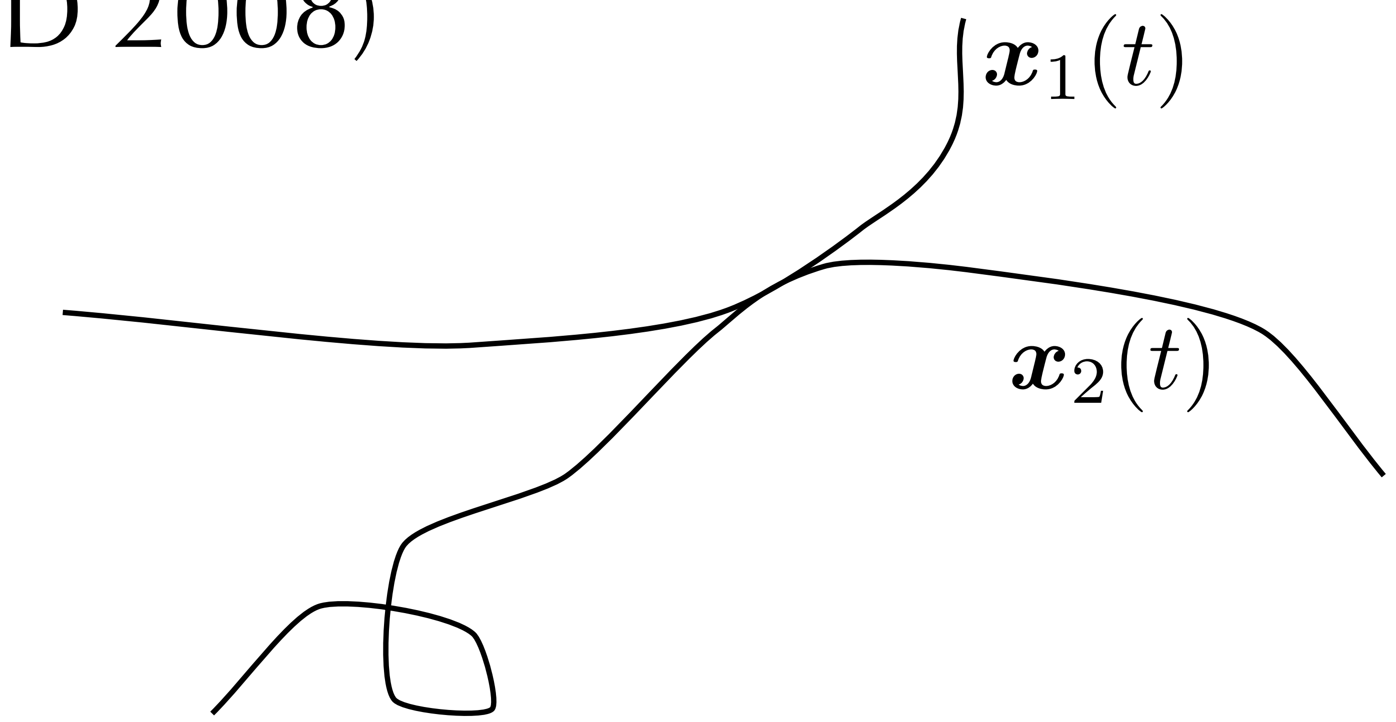
- Diffusion is not the unique source of randomness

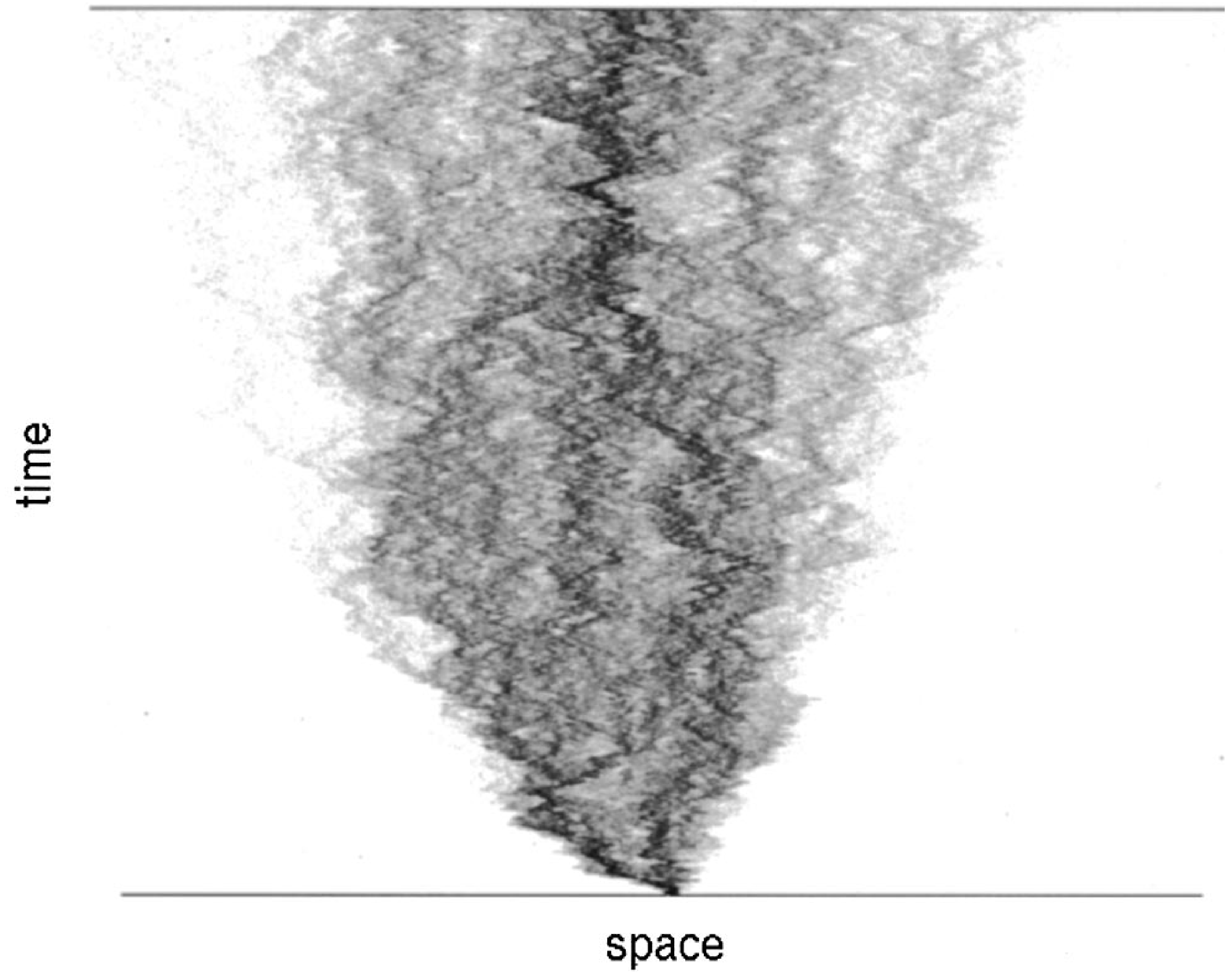
(Bernard *et al.*, *J. Stat. Phys.* 1998; Eyink, *Physica* D 2008)

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x},t)$$

$$|\boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x}',t')| \sim |\boldsymbol{x} - \boldsymbol{x}'|^h$$

$h = 1/3 < 1$ ⇒ not Lipschitz ⇒ non-uniqueness





$$\kappa \to 0, \ \ \nu \to 0$$

$$p_2(\boldsymbol{x}_1, \boldsymbol{x}_2, t \,|\, \boldsymbol{x}_0, \boldsymbol{x}_0, 0) \neq \delta^d(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$

Turbulent mixing is infinitely more efficient than any chaotic flow!

$$|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)| \simeq |\boldsymbol{x}_1(0) - \boldsymbol{x}_2(0)|\, \mathrm{e}^{\lambda\, t}$$

Dissipative anomaly

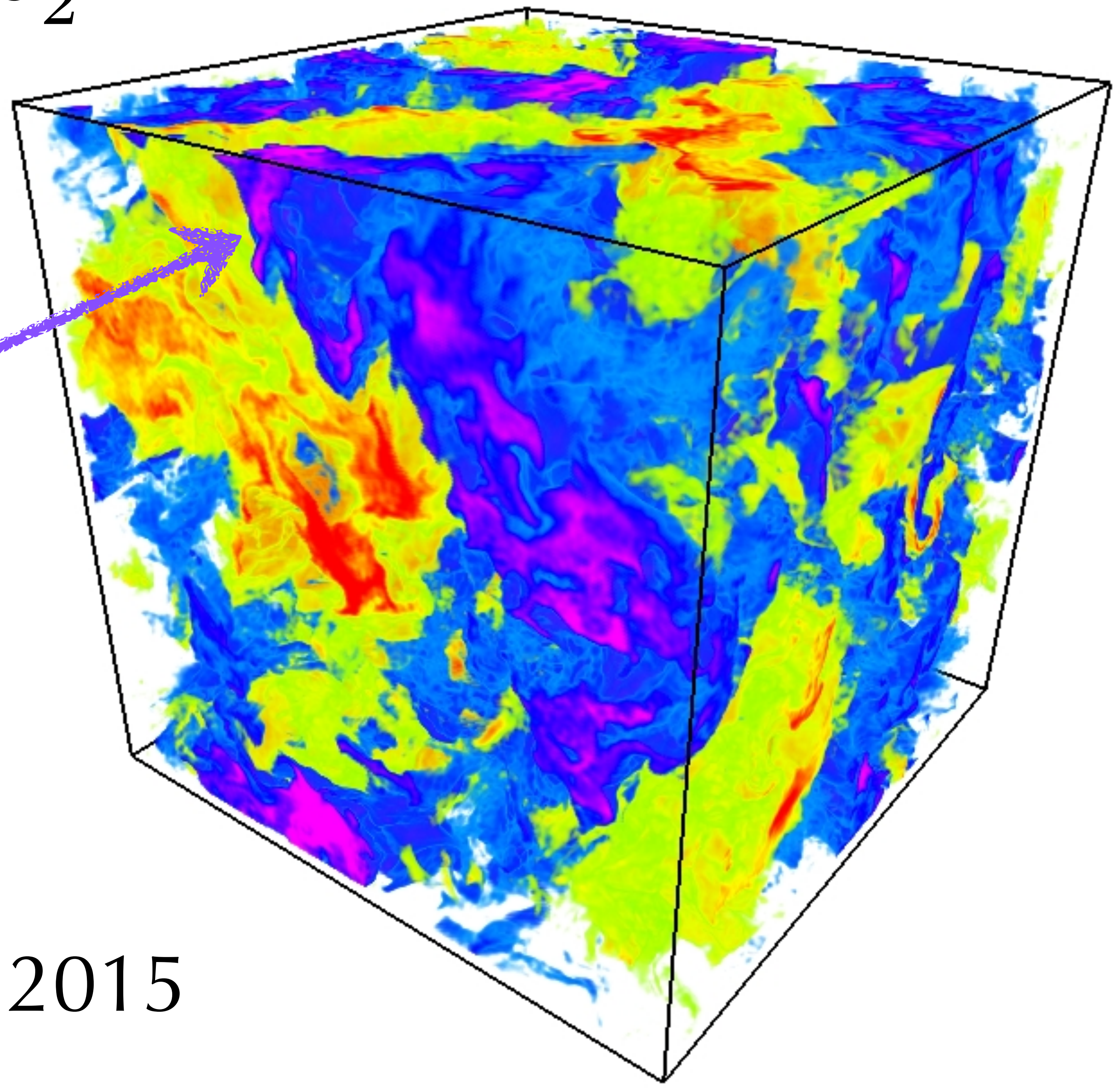
- Scalar dissipation

Larchevêque & Lesieur, *J. Méc.* 1981
Nelkin & Kerr, *PoF* 1981 ; Thomson, *JFM* 1996

$$\varepsilon_\theta = -\kappa\langle(\nabla\theta)^2\rangle \to const \quad \text{when} \quad \kappa,\nu \to 0 \quad \text{with fixed} \quad Pr$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\theta(\boldsymbol{x},t)^2\rangle = \iint \langle\theta_0(\boldsymbol{x}_1^0)\theta_0(\boldsymbol{x}_2^0)\rangle \times \partial_t p_2(\boldsymbol{x},\boldsymbol{x},t|\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)\,\mathrm{d}\boldsymbol{x}_1^0\mathrm{d}\boldsymbol{x}_2^0$$

Backward motion



Fronts

- Relation with the turbulent anomalous dissipation of kinetic energy?

Burgers equation: Eyink & Drivas, *J. Stat. Phys.* 2015

Pair dispersion: ballistic regime

- Statistics of the two-point motion $\boldsymbol{R}(t) = \boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)$

  $\langle \cdot \rangle_{r_0}$ conditioned on a fixed initial distance $|\boldsymbol{R}(0)| = r_0$

- **Batchelor regime**

  Batchelor, *Proc. Camb. Phil. Soc.* 1952

  Short-time expansion: $\boldsymbol{R}(t) = \boldsymbol{R}(0) + t\,\delta\boldsymbol{u} + \frac{t^2}{2}\delta \mathrm{D}_t\boldsymbol{u} + \mathrm{O}(t^3)$

  $\delta\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}_1(0),0) - \boldsymbol{u}(\boldsymbol{x}_2(0),0),\ D_t\boldsymbol{u} = \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}$

$$\left\langle|\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2\right\rangle_{r_0} = t^2 S_2(r_0) + t^3\left\langle\delta\boldsymbol{u}\cdot\delta \mathrm{D}_t\boldsymbol{u}\right\rangle + \mathrm{O}(t^4)$$

$$S_2(r_0) = \left\langle|\delta\boldsymbol{u}|^2\right\rangle \sim (\varepsilon\, r_0)^{2/3} \qquad \langle\delta\boldsymbol{u}\cdot D_t\boldsymbol{u}\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle|\delta\boldsymbol{u}|^2\right\rangle = -2\,\varepsilon$$

- Crossover time $t_0 = \dfrac{S_2(r_0)}{2\varepsilon}$

  Ballistic separation for $t \ll t_0 \sim \varepsilon^{-1/3} r_0^{2/3}$

Richardson–Obukhov law

- Behavior for times larger than t_0

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \delta\boldsymbol{u}(\boldsymbol{R}) \sim (\varepsilon|\boldsymbol{R}|)^{1/3}$$

Explosive separation

$$\left\langle|\boldsymbol{R}(t)|^2\right\rangle_{r_0} \sim g\,\varepsilon\,t^3$$

for $t_0 \ll t \ll T_L$

Independent of the initial separation r_0

Richardson, *Proc. Roy. Soc. Lond.* 1926
Obukhov, *Izv. Akad. Nauk SSSR* 1941

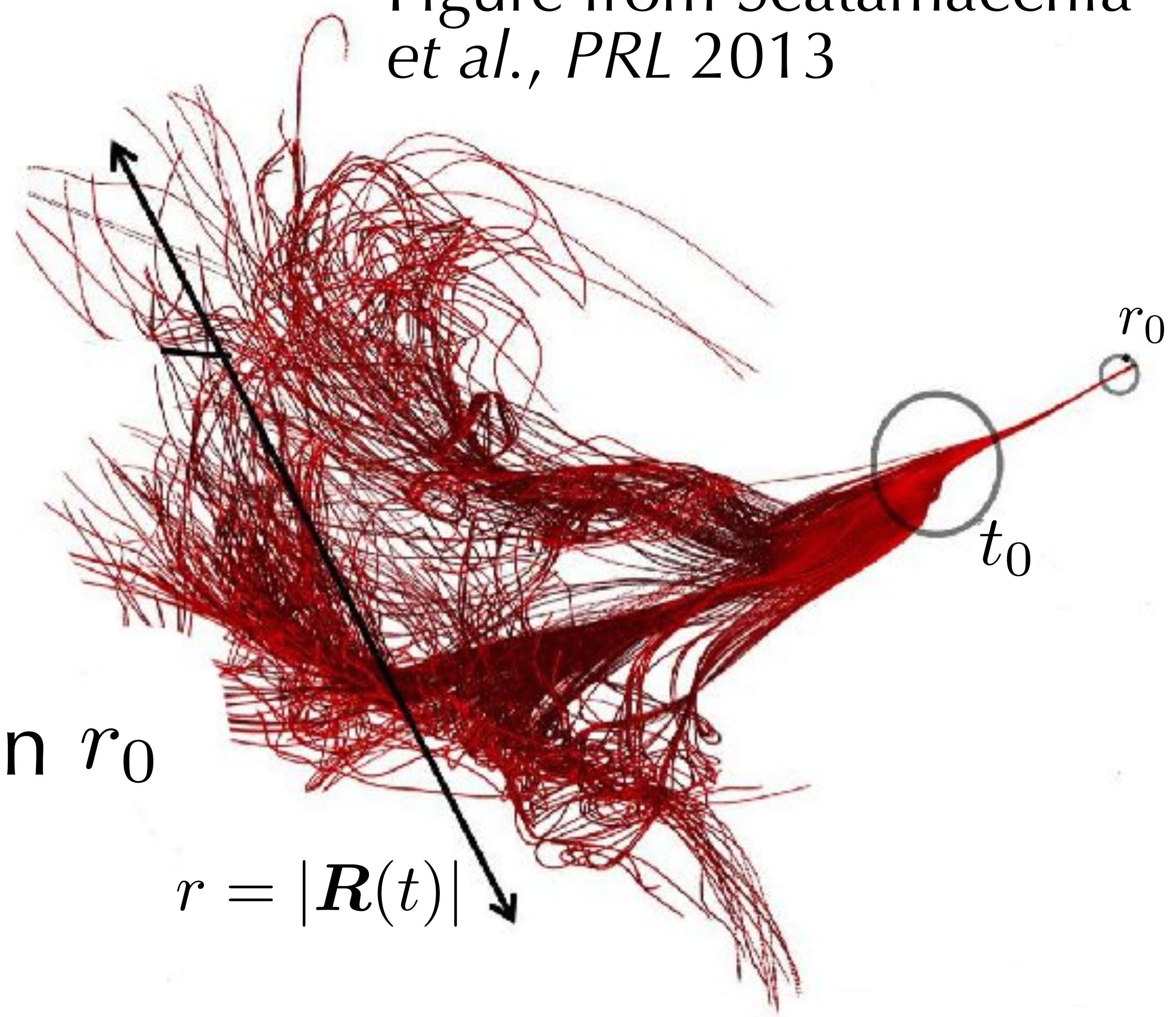


Figure from Scatamacchia *et al.*, *PRL* 2013

- Scaling regime?

$r \sim \varepsilon^{1/2}\,t^{3/2}$ suggests $p_2(r, t\,|\,r_0, 0) \sim \dfrac{1}{\varepsilon^{1/2}\,t^{3/2}}\,\Psi\!\left(\dfrac{r}{\varepsilon^{1/2}\,t^{3/2}}\right)$ for $t \gg t_0$

Difficult to observe numerically and experimentally because of the large temporal scale separation that is required: $\tau_\eta \ll t_0 \ll t \ll T_L$

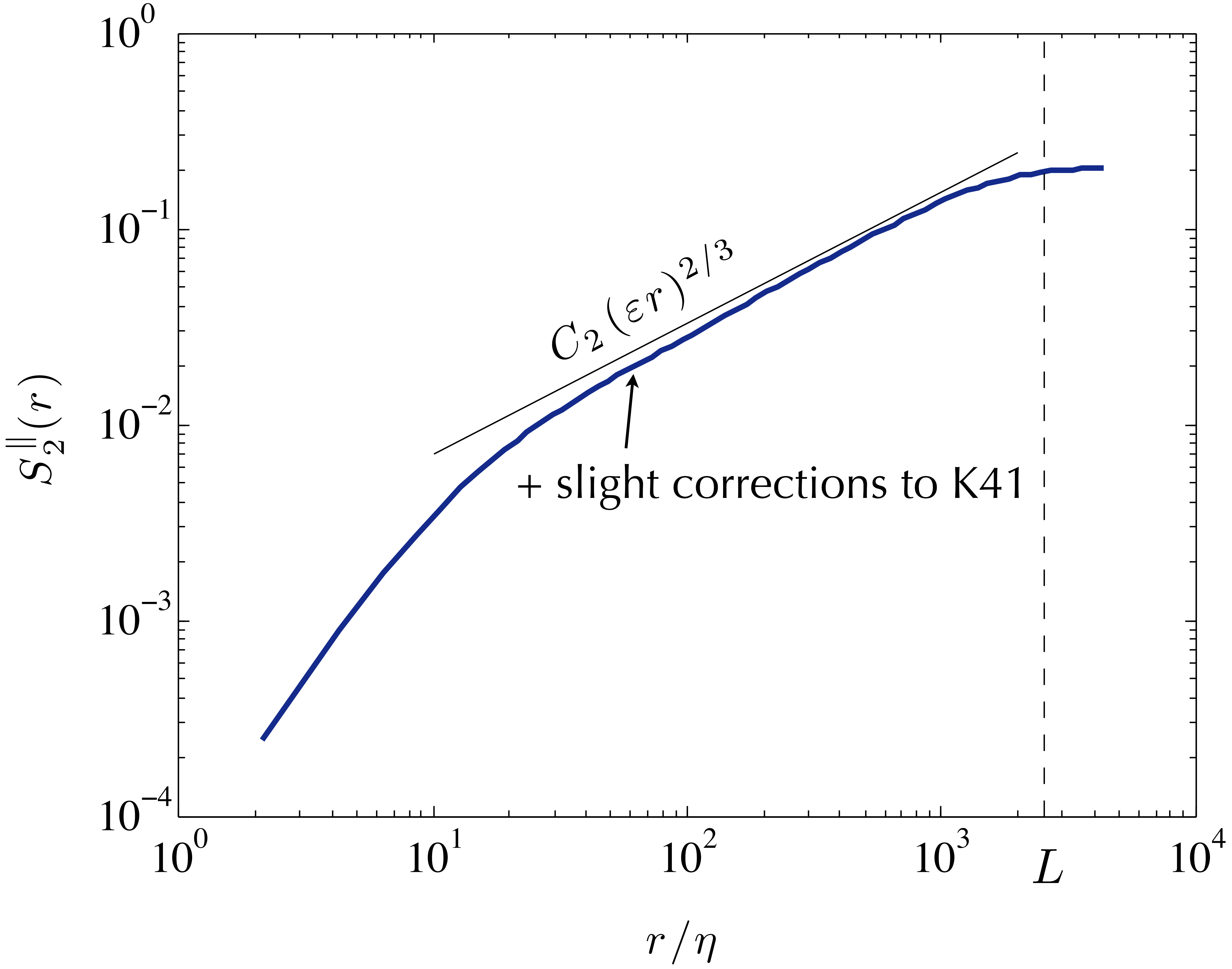
Review by Salazar & Collins
Ann. Rev. Fluid Mech. 2009

⇒ sub-leading terms? Mechanisms?

Numerics

LaTu: MPI pseudo-spectral solver (Homann et al. 2007)

R_λ	ν	η	τ_η	L	T_L	N^3
730	10^{-5}	$7.2 \cdot 10^{-4}$	0.05	1.85	9.6	4096^3



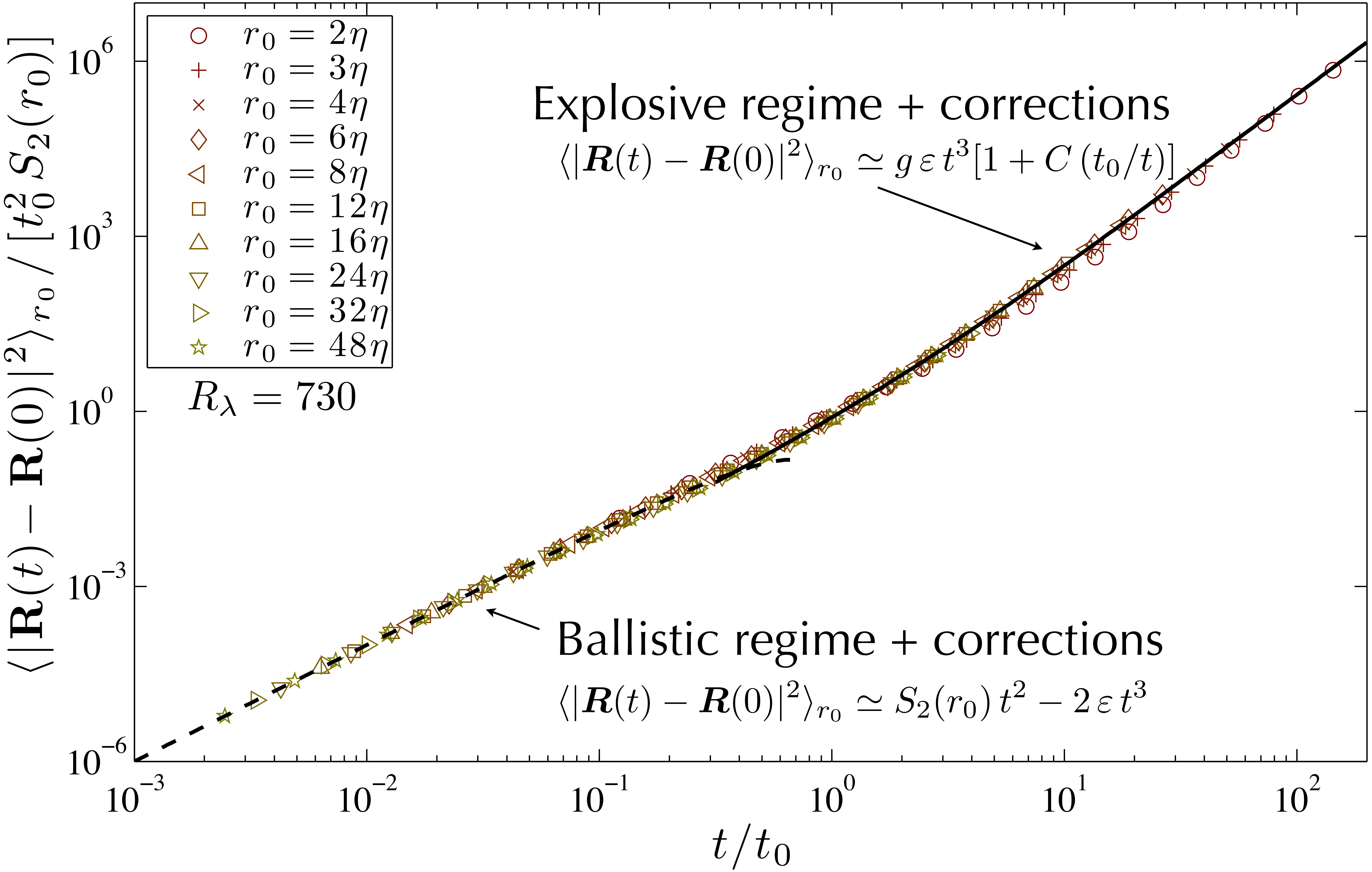
Incompressible NS + large-scale forcing

+ 10^7 Lagrangian trajectories

65 536 processes on BlueGene/P

JÜLICH FORSCHUNGSZENTRUM

Transition Ballistic/Explosive



Explosive regime + corrections

$$\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^2\rangle_{r_0} \simeq g\,\varepsilon\,t^3[1+C\,(t_0/t)]$$

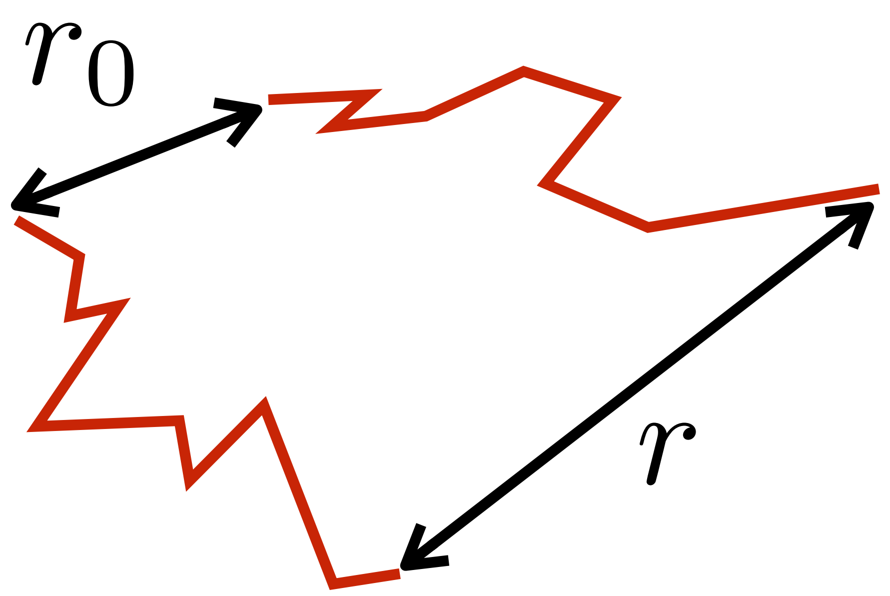
Ballistic regime + corrections

$$\langle|\boldsymbol{R}(t)-\boldsymbol{R}(0)|^2\rangle_{r_0} \simeq S_2(r_0)\,t^2 - 2\,\varepsilon\,t^3$$

Bitane *et al.*, *PRE* 2013

Richardson diffusion

Assumption: velocity differences **uncorrelated** ⇒ separation diffuses



Transition probability $p_2(r,t|r_0,0)$

$$\partial_t p_2 = \nabla\cdot(K(r)\nabla p_2)$$

+ K41(Obukhov) $K(r)\sim\varepsilon^{1/3}r^{4/3}$

⇒ $$p_2(r,t|r_0,0)\propto\frac{r^2}{t^{9/2}}\,\mathrm{e}^{-C\,r^{2/3}/(\varepsilon\,t)}$$ and $$\left\langle|\boldsymbol{R}(t)|^2\right\rangle_{r_0}\sim g\,\varepsilon\,t^3$$

Formalized for Kraichnan model: Gaussian velocity with correlation

$$\langle u^i(\boldsymbol{x},t)\,u^j(\boldsymbol{x}',t')\rangle = \delta(t-t')\,[2D_0\delta^{ij}-d^{ij}(\boldsymbol{x}-\boldsymbol{x}')]$$

$$d^{ij}(\boldsymbol{r}) = D_1\,r^{\xi}[(d-1+\xi)\,\delta^{ij}-\xi\,r^ir^j/r^2]$$

see Falkovich,
Gawedzki, Vergassola,
Rev. Mod. Phys. 2001

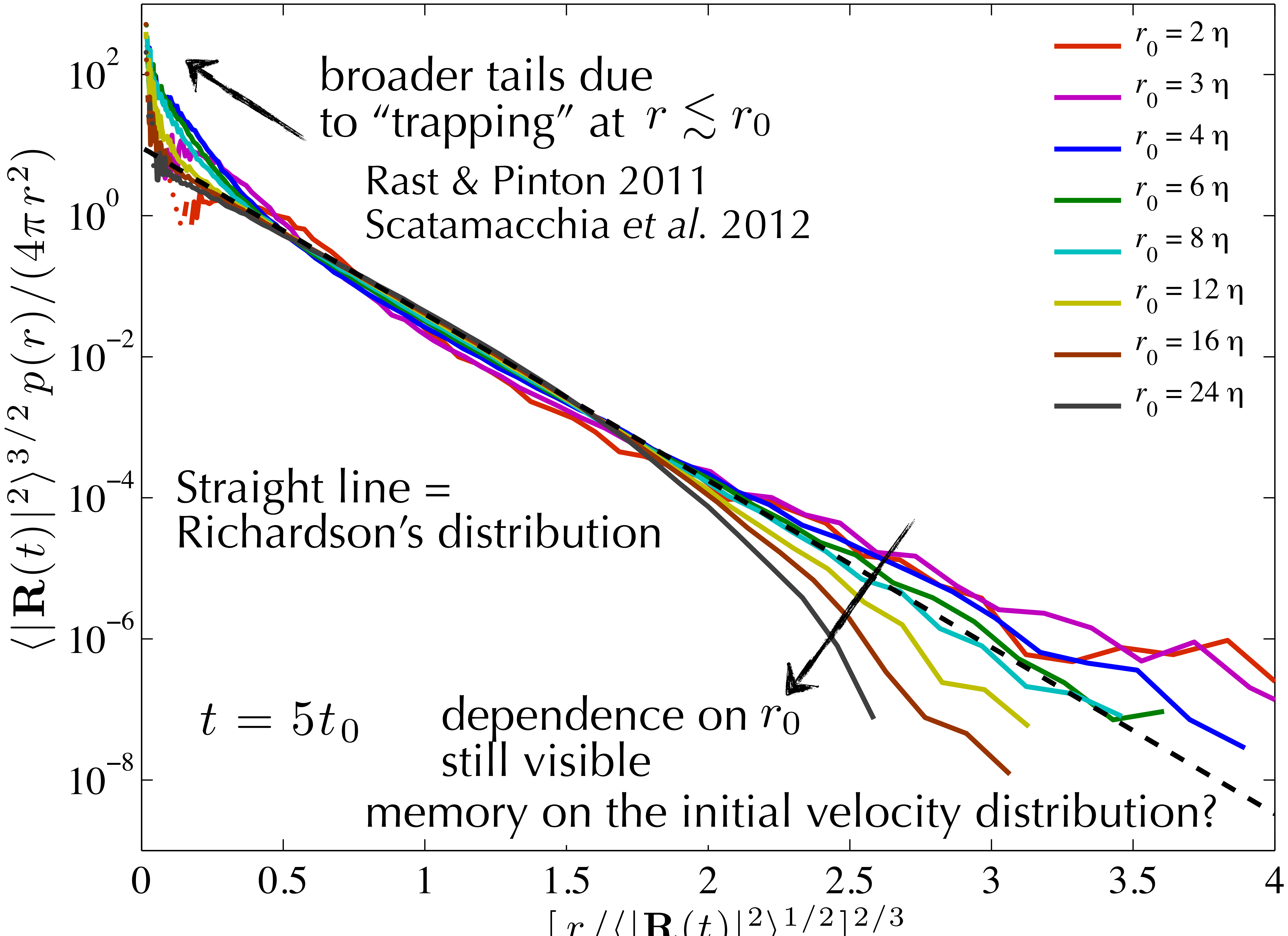
Shortcoming: velocity difference get uncorrelated on times $\mathrm{O}(t)$

Phenomenology ⇒ correlation time $\tau_r\sim r^{2/3}$

$$+r^2\sim t^3\;\Rightarrow\;\tau_r\sim t\;\ldots$$

Distribution of distances

Comparison to Richardson's distribution $p_2(r,t|r_0,0) \propto \frac{r^2}{t^{9/2}}\,\mathrm{e}^{-C\,r^{2/3}/(\varepsilon\, t)}$



broader tails due to "trapping" at $r \lesssim r_0$

Rast & Pinton 2011
Scatamacchia *et al.* 2012

Straight line = Richardson's distribution

$t = 5t_0$

dependence on r_0 still visible

memory on the initial velocity distribution?

Such a representation emphasizes the collapse of the core of the distribution…

Markov models

Assumption: acceleration differences are **short correlated**

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{A} = \delta \mathrm{D}_t \boldsymbol{u} \longleftarrow$$ components correlated over a time $\mathrm{O}(\tau_\eta)$

Central-Limit Theorem: $\boldsymbol{A} \overset{\text{law}}{\equiv} \sqrt{\tau_\eta}\, \mathbb{A}(\boldsymbol{R}, \boldsymbol{V}) \circ \boldsymbol{\eta}(t)$ when $t \gg \tau_\eta$

with $\mathbb{A}^{\mathsf{T}}\mathbb{A} = \langle \delta \mathrm{D}_t \boldsymbol{u} \otimes \delta \mathrm{D}_t \boldsymbol{u} \,|\, \delta \boldsymbol{u} \rangle$ correlations of acceleration differences conditioned on $\delta \boldsymbol{u}$

General form:
$$\begin{cases} \mathrm{d}\boldsymbol{R} = \boldsymbol{V}\, \mathrm{d}t \\ \mathrm{d}\boldsymbol{V} = \boldsymbol{a}(\boldsymbol{R}, \boldsymbol{V}, t)\mathrm{d}t + \mathbb{B}(\boldsymbol{R}, \boldsymbol{V}, t)\, \mathrm{d}\boldsymbol{W} \end{cases}$$

Kurbanmuradov & Sabelfeld (1995); Sawford (2001)

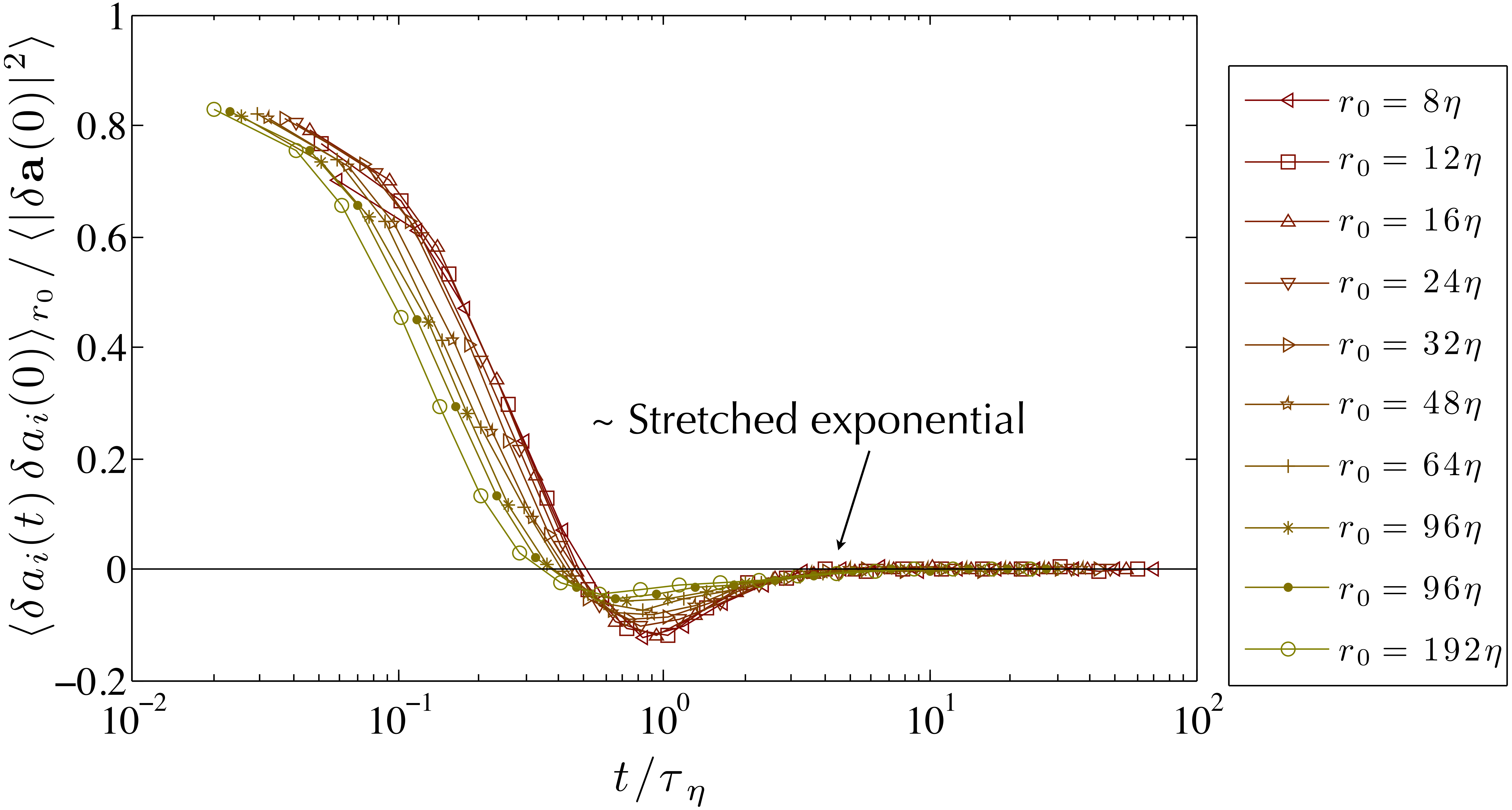
⇒ Fokker–Planck equation for $p(\boldsymbol{r}, \boldsymbol{v}, t | \boldsymbol{r}_0, \boldsymbol{v}_0, 0)$

$$\partial_t p + \partial_{r_i}(v_i\, p) + \partial_{v_i}(a_i\, p) = \frac{1}{2}\partial_{v_i}\partial_{v_j}[B_{ik}B_{jk}\, p]$$

Admissibility condition: "well-mixing"

Consistency with Eulerian statistics $p_E(\boldsymbol{r}, \boldsymbol{v})$ is a stationary solution associated to an initial uniform distribution in space (Thomson 1991)

Time-correlation of acceleration



~ Stretched exponential

Bitane *et al.*, *JoT* 2013

Limits of Markov modeling

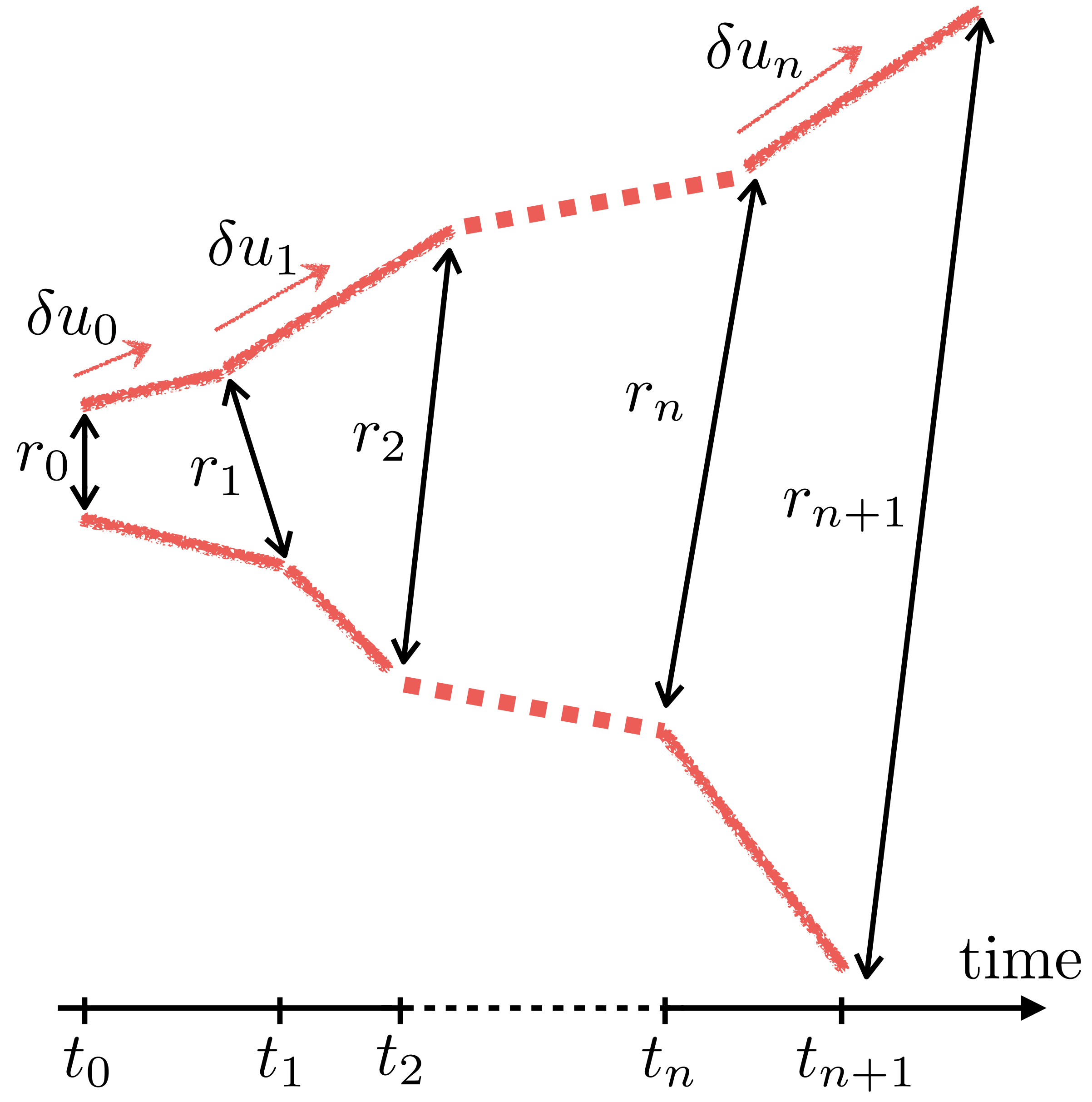
- Is acceleration really short-time correlated?
  ⇒ OK for components but not amplitude (Mordant *et al.*, *PRL* 2004)
  ⇒ Stretched exponential correlations (non-mixing process)

- Most models lead to an asymptotic diffusion of velocities.
  Is this the mechanism explaining Richardson's scaling $R \sim t^{3/2}$?
  ⇒ Is it compatible with the observed intermittent behaviors?
  e.g. exit times (Boffetta *et al.*, *PRE* 1999; Boffetta & Sokolov, *PRL* 2002)
  ⇒ Are finite-*Re* effects solely responsible for lack of scaling?
  (Scatamacchia *et al.*, *PRL* 2012)

- Is turbulent relative motion really a Markov process?
  ⇒ Relation to Lévy walks / waiting times approaches
  (Shlesinger *et al.*, *PRL* 1987; Faller, *JFM* 1996; Rast & Pinton, *PRL* 2011)
  ⇒ Some deviations might be due to memory effects
  (Ilyin *et al.*, *PRE* 2010; Eyink & Benveniste, *PRE* 2013)

A piecewise-ballistic scenario

- Ballistic regime is key in the convergence to the explosive behavior
- Build a simple model that reproduces some essential mechanisms



Continuous-Time Random Walk

$$\vec{r}_n \mapsto \vec{r}_{n+1} = \vec{r}_n + \Delta t_n \delta \vec{u}_n$$
$$t_n \mapsto t_{n+1} = t_n + \Delta t_n$$

$$\begin{cases} \delta u_n \text{ and } \Delta t_n \text{ depend on } r_n \\ \text{the } \delta u_n\text{'s are independent from each other} \end{cases}$$

non-Markovian w.r.t. to the continuous time

K41 version: $\begin{cases} \delta u_n \sim (\varepsilon\, r_n)^{1/3} \\ \Delta t_n \sim r_n/\delta u_n \sim \varepsilon^{-1/3} r_n^{2/3} \end{cases}$

$$\left.\begin{array}{l} r_{n+1} \simeq r_n + X_n\, r_n \quad \Rightarrow \ln(r_n/r_0) \simeq \sum_k \ln(1+X_k) \propto n \\ t_{n+1} \simeq t_n + Y_n\, r_n^{2/3} \quad \Rightarrow t_n \simeq \sum_k Y_n\, r_n^{2/3} \propto \mathrm{e}^{\frac{2}{3}n} \end{array}\right\} \Rightarrow \ln(r_t/r_0) \simeq \frac{3}{2} \ln(t/t_0)$$

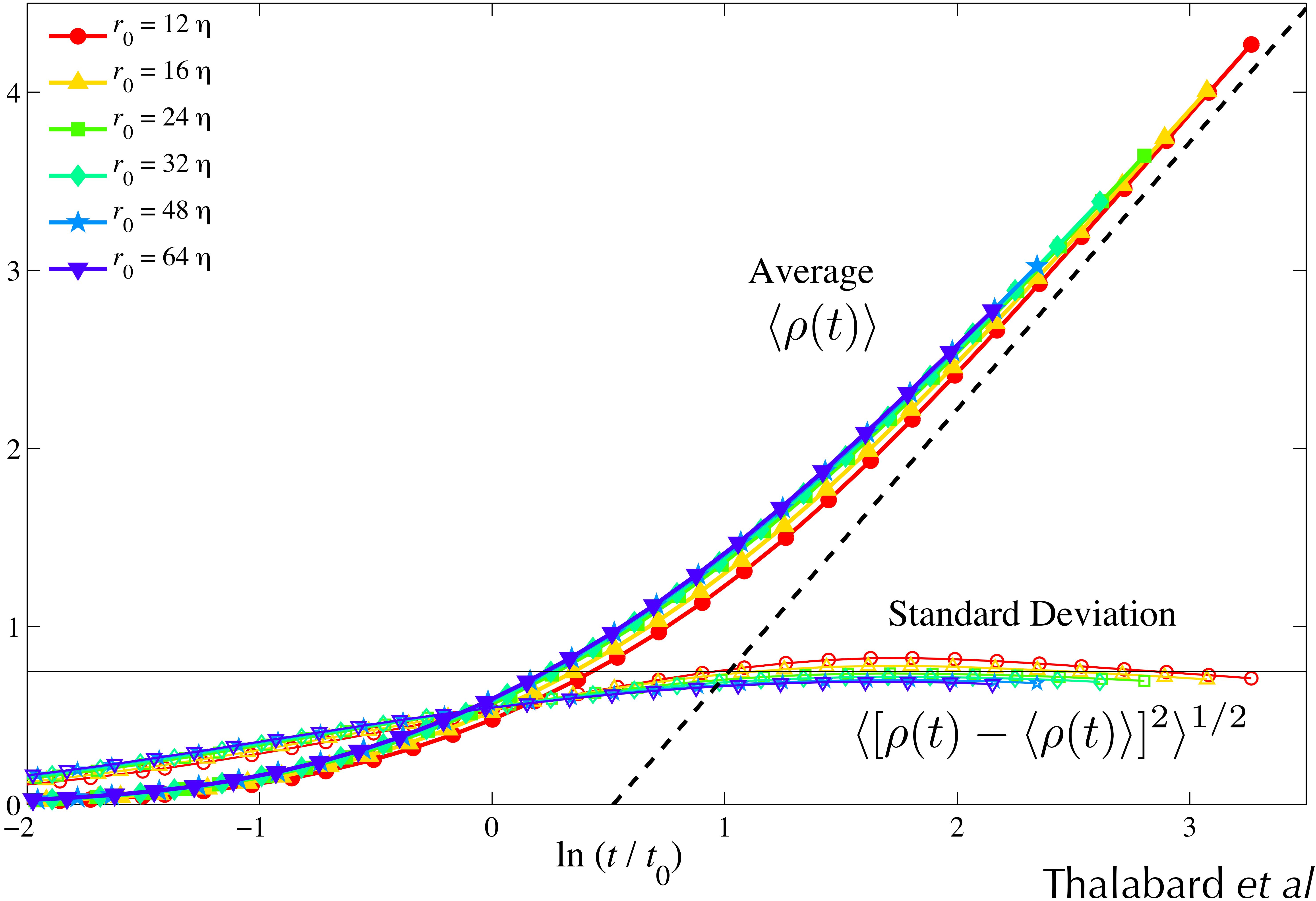
- Is $\ln(|\boldsymbol{R}(t)|/r_0)$ a **self-averaging quantity**?
  Law of large numbers? Central-limit theorem? Large deviations?

Are distances a multiplicative process?

- The ballistic scenario suggests $\rho = \ln(|\boldsymbol{R}(t)|/r_0)$ as a relevant quantity

Richardson's distribution: $\langle \rho(t) \rangle = (3/2)\ln(t/t_0) + (1/2)\ln g - 0.46$

$$\langle [\rho(t) - \langle \rho(t) \rangle]^2 \rangle^{1/2} = 0.748$$



Thalabard *et al. JFM* 2014

Further modeling

Time increment: dissipation time $\Delta t_n = |\delta\vec{u}_n|^2/\varepsilon$

$$\alpha_n = \delta u_n^{\parallel}/|\delta\vec{u}_n|$$
$$\beta_n = |\delta\vec{u}_n|^3/(\varepsilon r_n)$$

with statistics independent of r_n (K41)

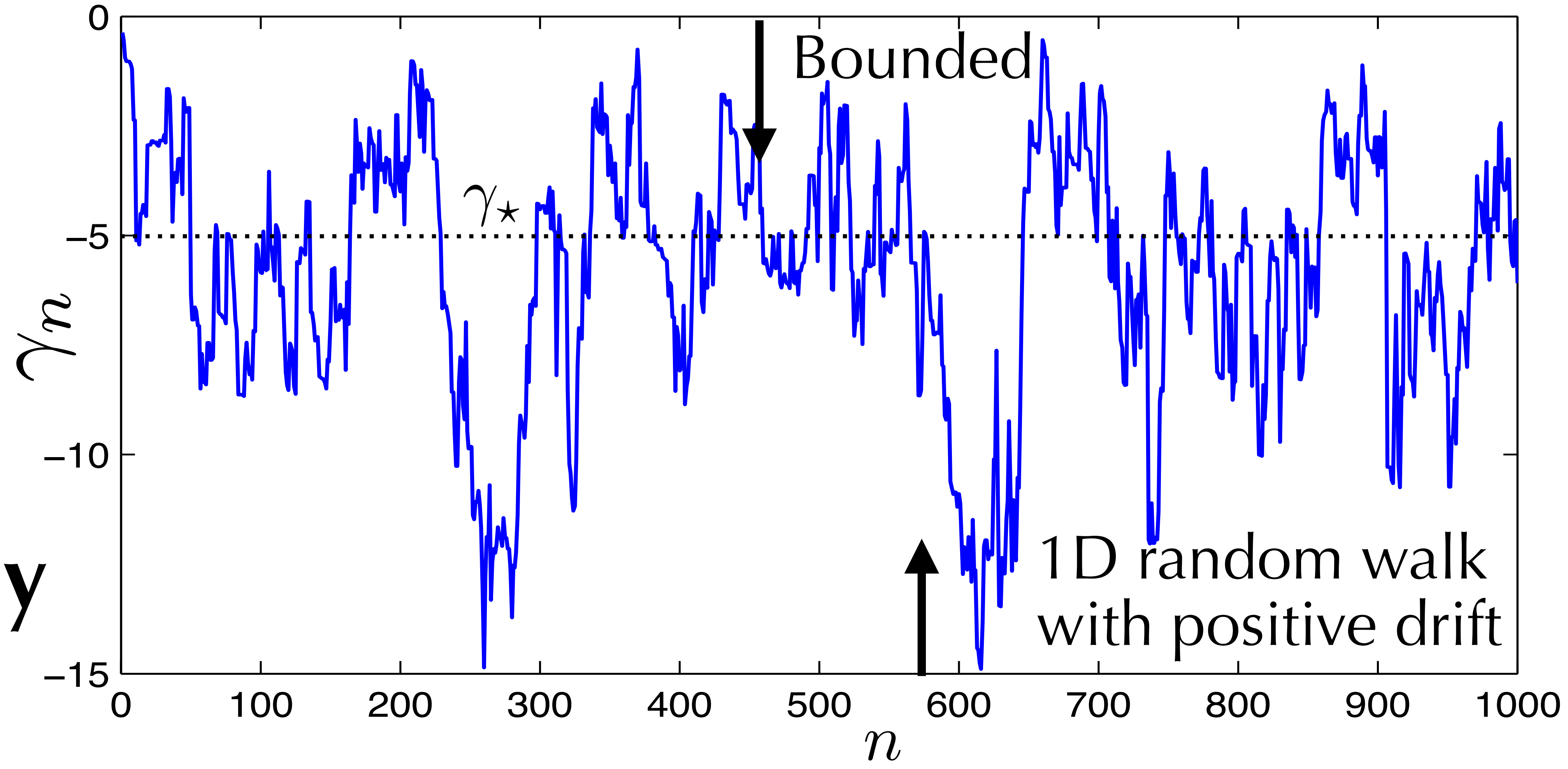
$$\begin{cases} r_{n+1} = r_n\sqrt{1+2\alpha_n\beta_n+\beta_n^2} \\ t_{n+1} = t_n + \varepsilon^{-1/3}\beta_n^{2/3}r_n^{2/3} \end{cases}$$

Change of variables: $\gamma_n = \ln(r_n/r_0) - (3/2)\ln(t/t_0)$ $\quad t_0 = \varepsilon^{-1/3}r_0^{2/3}$

$$\gamma_{n+1} = \gamma_n + \frac{3}{2}\ln\frac{(1+2\alpha_n\beta_n+\beta_n^2)^{1/3}}{1+\beta_n^{2/3}\mathrm{e}^{(2/3)\gamma_n}}$$

Recurrence point $\gamma_\star$

⇒ the γ_n's are becoming **stationary**

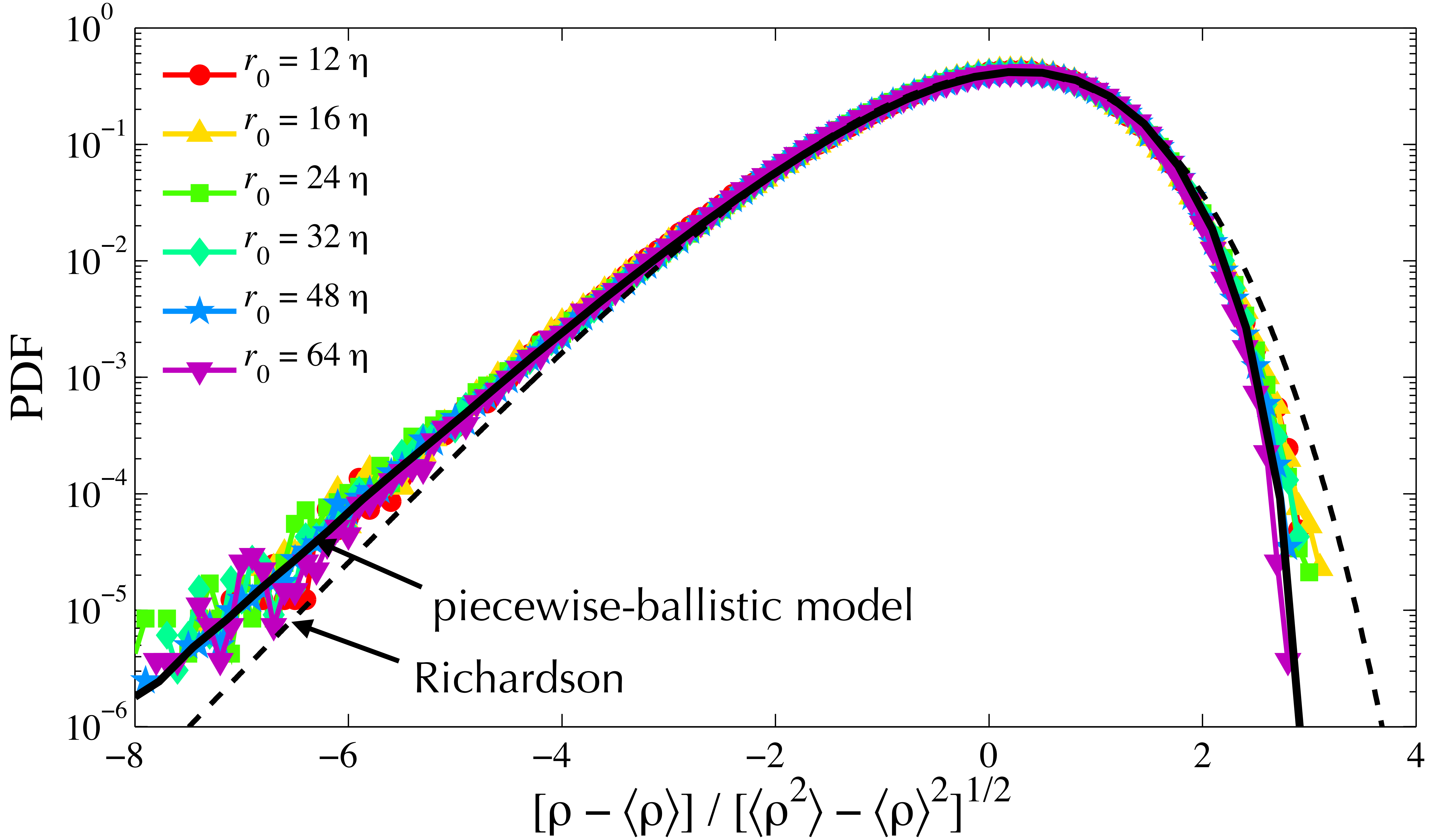


This suggests for $\rho = \ln(|\boldsymbol{R}(t)|/r_0)$

$\langle\rho\rangle \simeq (3/2)\ln(t/t_0) + \langle\gamma\rangle$ $\quad \mathrm{Var}\,[\rho] \simeq \mathrm{Var}\,[\gamma] = \mathrm{const}$ $\quad \mathrm{PDF}\,(\rho) \simeq \Psi(\rho - \langle\rho\rangle)$

Distribution of the log-separation

Scale invariance for the distribution of $\rho = \ln(|\boldsymbol{R}(t)|/r_0)$



The collapsing distribution can be reproduced by properly choosing the distribution of $\alpha_n = \delta u_n^{\parallel}/|\delta\vec{u}_n|$ and $\beta_n = |\delta\vec{u}_n|^3/(\varepsilon r_n)$

Open questions

$$\begin{cases} r_{n+1} = r_n \sqrt{1 + 2\alpha_n \beta_n + \beta_n^2} \\ t_{n+1} = t_n + \varepsilon^{-1/3} \beta_n^{2/3} r_n^{2/3} \end{cases} \qquad \begin{aligned} \alpha_n &= \delta u_n^{\|} / |\delta \vec{u}_n| \\ \beta_n &= |\delta \vec{u}_n|^3 / (\varepsilon r_n) \end{aligned}$$

Effect of the fluid velocity intermittency

How is the scaling behavior affected when K41 is not fulfilled?

⇒ Studying extensions of the model assuming multifractal statistics

e.g. $\beta_n \propto r_n^{3h_n - 1}$with $p(h_n) \propto r_n^{3-D(h_n)}$

⇒ Is the long-time behavior still following a scaling regime?

Time irreversibility

Relative dispersion is faster backward in time than forward

What are the underlying mechanisms? How to quantify?

⇒ In the model, the only symmetry-breaking quantity is α_n

How is the “Richardson constant” altered when $\alpha_n \mapsto -\alpha_n$?

The model might not be enough to address this issue:

in real flows, $\alpha$and $\beta$are correlated

Lecture 2: Anomalous scaling

Summary of lecture 1

2-point motion / 2nd-order statistics in the "explosive regime" pretty well described by **Richardson–Obukhov scaling**:

$$r \sim \varepsilon^{1/2}\, t^{3/2} \qquad p_2(r,t\,|\,r_0,0) \sim \frac{1}{\varepsilon^{1/2}\, t^{3/2}}\, \Psi\!\left(\frac{r}{\varepsilon^{1/2}\, t^{3/2}}\right)$$

⇒ Possible intermittent corrections?

$$p_2(r,t\,|\,r_0,0) \sim \frac{1}{\varepsilon^{1/2} t^{3/2}}\, \bar{\Psi}\!\left(\frac{r}{\varepsilon^{1/2} t^{3/2}}, \boldsymbol{\frac{r}{L}}\right) \quad \text{e.g.} \quad \bar{\Psi} = \left(\frac{r}{L}\right)^{\alpha} \Psi\!\left(\frac{r}{\varepsilon^{1/2} t^{3/2}}\right)$$

Origin? Not turbulent transport itself but maybe fluid velocity anomalous scaling

Second lecture:

⇒ n-point motion / higher-order statistics is intrinsically intermittent (Kraichnan flow)

⇒ A concrete example where this matters

Passive scalar intermittency

Structure functions of a passive scalar

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + \phi$$

$$\delta\theta = \theta(\boldsymbol{x} + \boldsymbol{r}, t) - \theta(\boldsymbol{x}, t)$$

$$\langle \delta\theta^q \rangle \sim r^{\zeta_q}$$

Exact relation (Yaglom 1949):

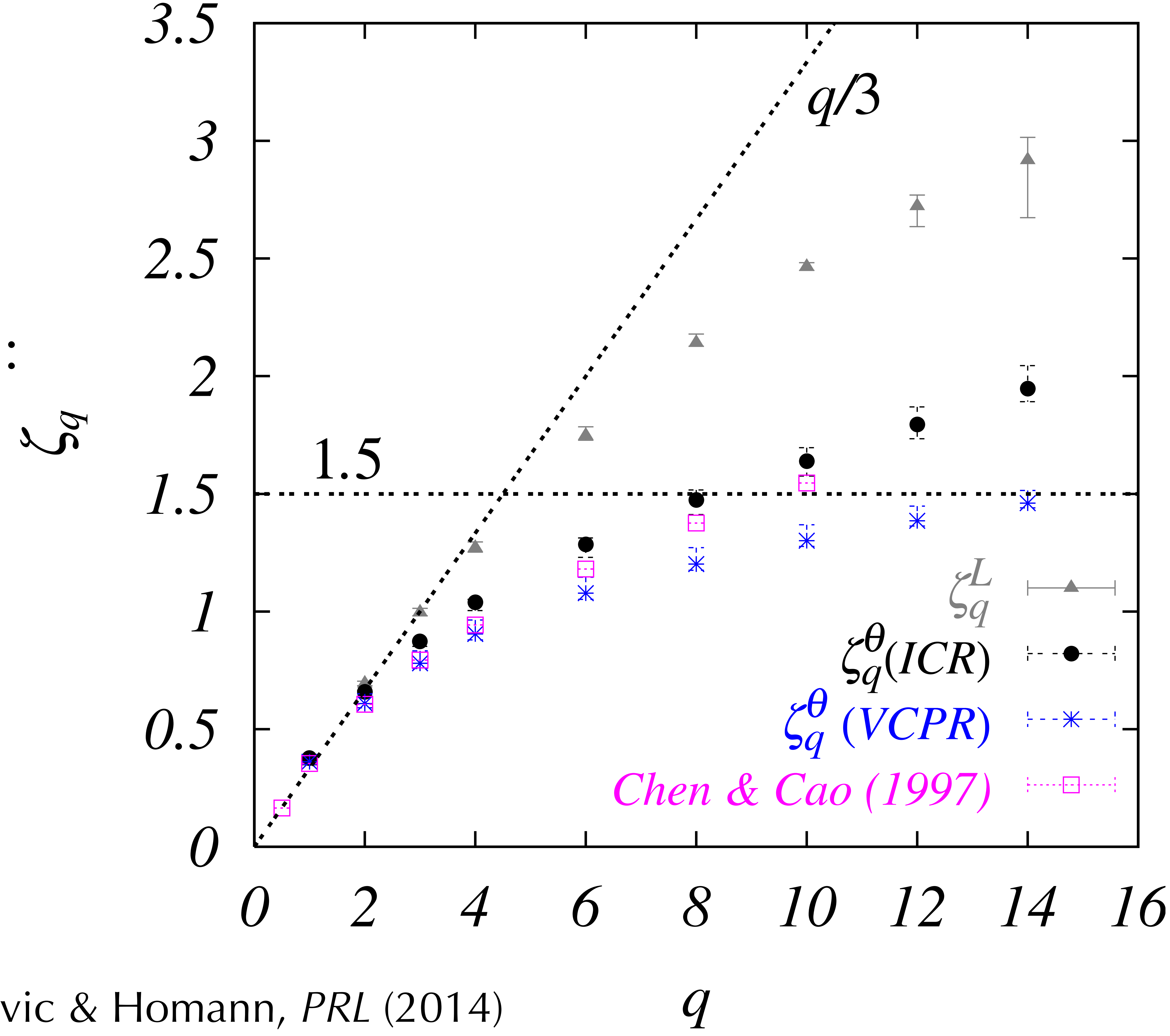
$$\left\langle \delta^{\parallel} u \, [\delta\theta]^2 \right\rangle = -\frac{4}{3}\, \varepsilon_\theta \, r$$

$$\delta^{\parallel} u = \hat{\boldsymbol{r}} \cdot \delta \boldsymbol{u} \qquad \varepsilon_\theta = \kappa \left\langle (\nabla\theta)^2 \right\rangle$$

Dimensional scaling (K41):

$$\zeta_q = q/3$$

Anomalous scaling



from Watanabe & Gotoh, *NJP* (2004)

see also Gotoh & Watanabe, *PRL* (2015); Bec, Krstulovic & Homann, *PRL* (2014)

Geometric structure of intermittency

Strong intermittency related to the presence of **“multifractal fronts”**

4096^3
$R_\lambda = 730$

$\delta_r\theta \propto r^h$

with prob. $\propto r^{d-D(h)}$

$\zeta_q = \inf_h [qh + d - D(h)]$

Lagrangian interpretation

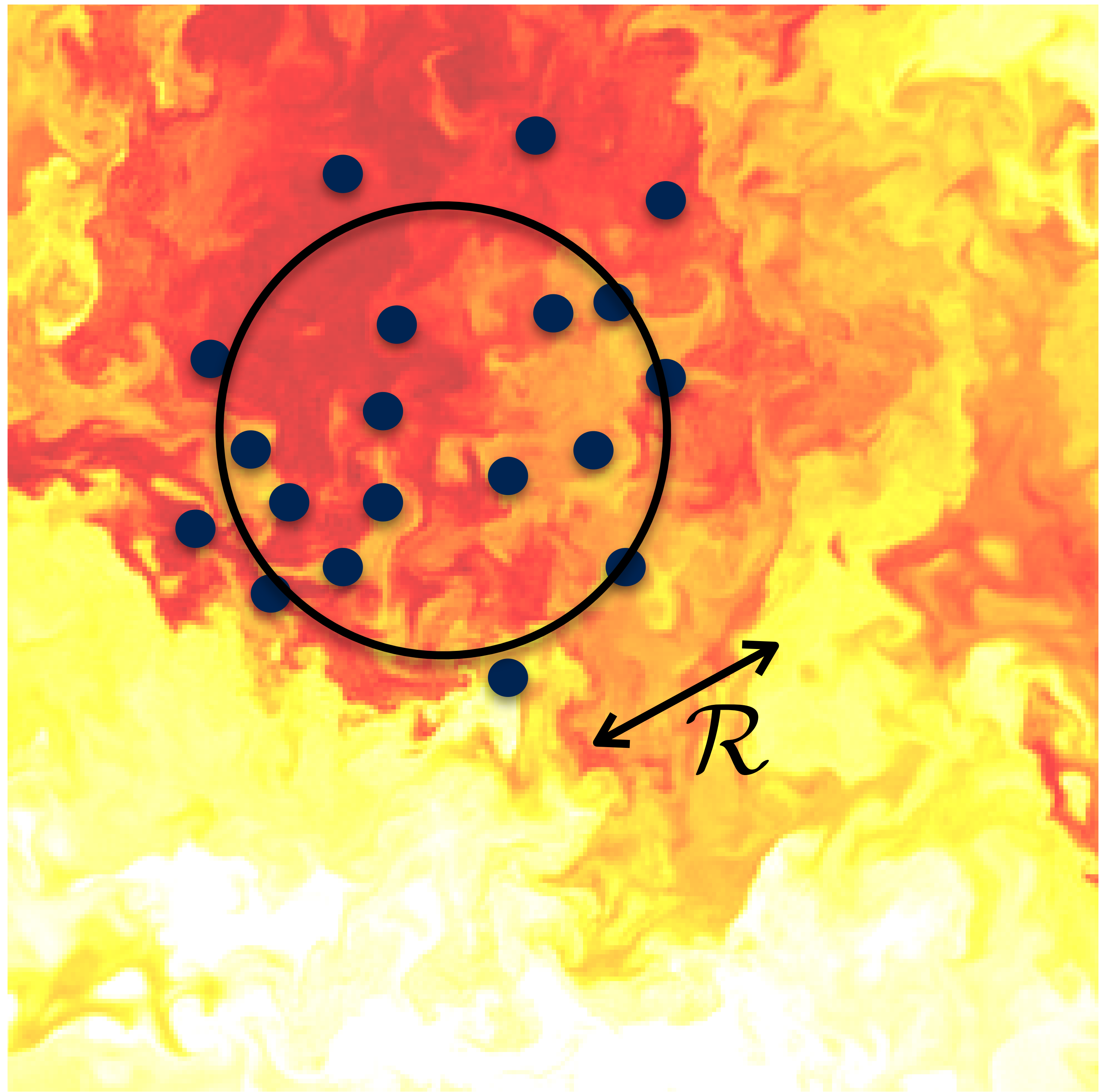
Lagrangian viewpoint

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t) = \boldsymbol{u}(\boldsymbol{X}(t),t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t)$$

$$\delta\theta^q \quad \longleftrightarrow \quad \boldsymbol{X}_1,\dots,\boldsymbol{X}_q$$

$$\bar{\boldsymbol{X}} = \frac{1}{q}\sum_n \boldsymbol{X}_n$$ single-particle motion

$$\tilde{\boldsymbol{X}}_n = \boldsymbol{X}_n - \bar{\boldsymbol{X}}$$



Time evolution of both **size** and **shape** of the cloud of tracers

$$\mathcal{R}^2 = \frac{1}{q}\sum_q |\tilde{\boldsymbol{X}}_n|^2$$

$$\hat{\boldsymbol{X}}_n = \tilde{\boldsymbol{X}}_n/\mathcal{R} \Longrightarrow \text{ shape}$$

Lessons from Kraichnan model

Gaussian velocity field, δ-correlated in time, self-similar in space

$$\langle u^i(\boldsymbol{x},t)\,u^j(\boldsymbol{x}',t')\rangle = \delta(t-t')\,[2D_0\delta^{ij} - d^{ij}(\boldsymbol{x}-\boldsymbol{x}')]$$

$$d^{ij}(\boldsymbol{r}) = D_1\,r^\xi[(d-1+\xi)\,\delta^{ij} - \xi\,r^i r^j/r^2]$$

$\xi = 4/3 \longleftrightarrow$ turbulence

q-point motion: $p_q(\tilde{\boldsymbol{X}}_1,\dots,\tilde{\boldsymbol{X}}_q,t\,|\,\boldsymbol{x}_1,\dots,\boldsymbol{x}_q,0)$

Backward Kolmogorov (≈ Fokker–Planck): $\partial_t p_q = \mathcal{M}_q\,p_q$

$$\mathcal{M}_q = \sum_{n<m}\left[d^{ij}(\boldsymbol{x}_n-\boldsymbol{x}_m) + 2\kappa\delta^{ij}\right]\partial_{x_n^i}\,\partial_{x_m^j}$$

There exists **zero modes** $\mathcal{M}_q\,f_q = 0$ with non-trivial scaling properties:

$$f_q(\lambda\boldsymbol{x}_1,\dots,\lambda\boldsymbol{x}_q) = \lambda^{\zeta_q} f_q(\boldsymbol{x}_1,\dots,\boldsymbol{x}_q)$$

Falkovich, Gawedzki, Vergassola, *Rev. Mod. Phys.* 2001

Lagrangian statistical conservation law

Zero modes are preserved by the dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathcal{R}^{\zeta_q} g_q(\hat{\boldsymbol{X}}_1,\ldots,\hat{\boldsymbol{X}}_q)\right\rangle = 0$$

size factor $\propto t^{3\zeta_q/2}$

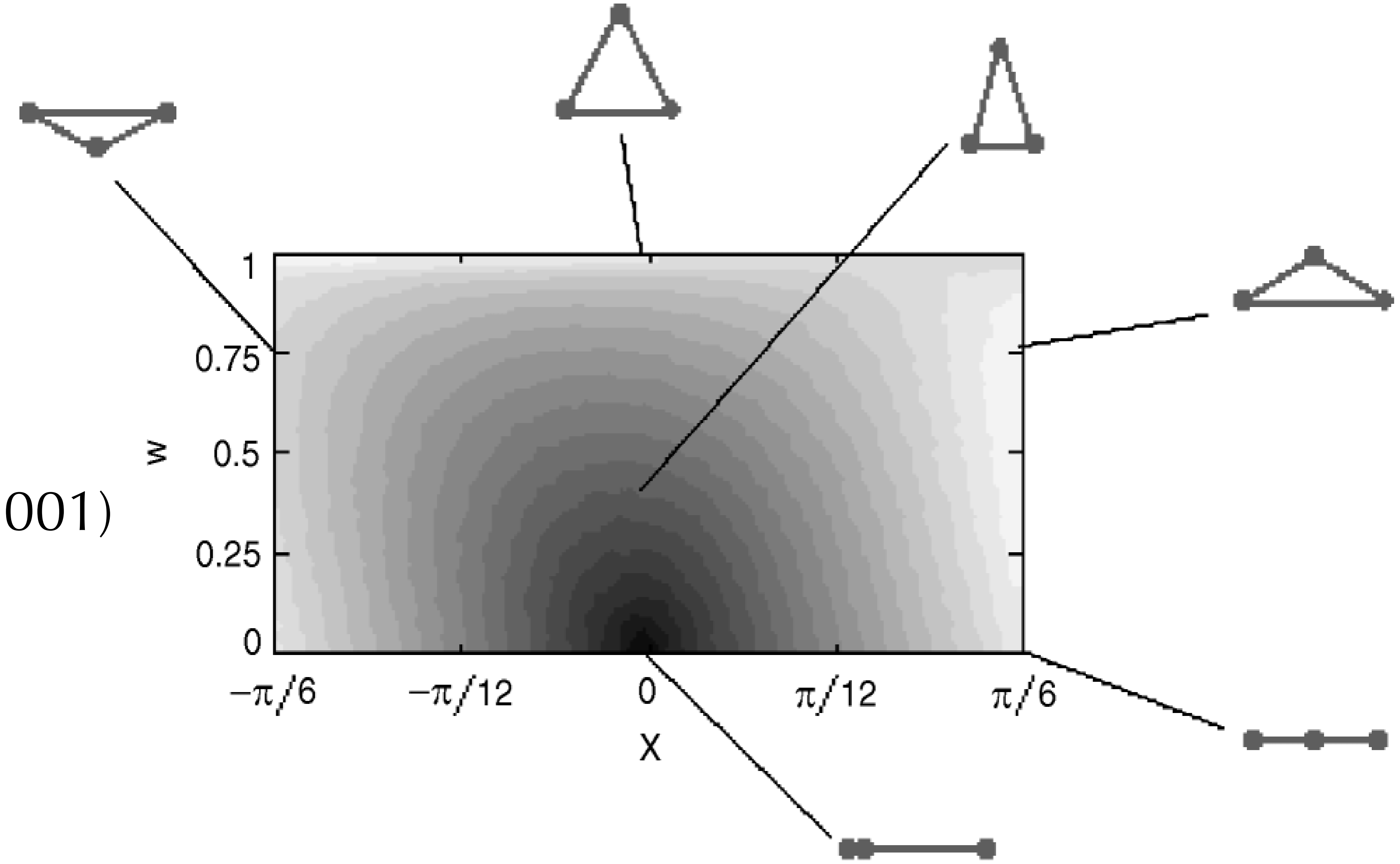
shape function

Bernard, Gawedzki, Kupiainen, *J. Stat. Phys.* (1997); Pumir, Shraiman, Chertkov, *PRL* (2000)

$q = 3$

2D Inverse cascade

Celani & Vergassola, *PRL* (2001)



$$p_3(R_1,R_2,t\,|\,r_1,r_2,0) \sim \left(\frac{R_1^2+R_2^2}{L^2}\right)^{\frac{\zeta_3-1}{2}} \frac{1}{\varepsilon t^3}\Psi\left(\frac{R_1}{\varepsilon^{1/2}t^{3/2}},\frac{R_2}{\varepsilon^{1/2}t^{3/2}}\right)$$

Intermittent correction

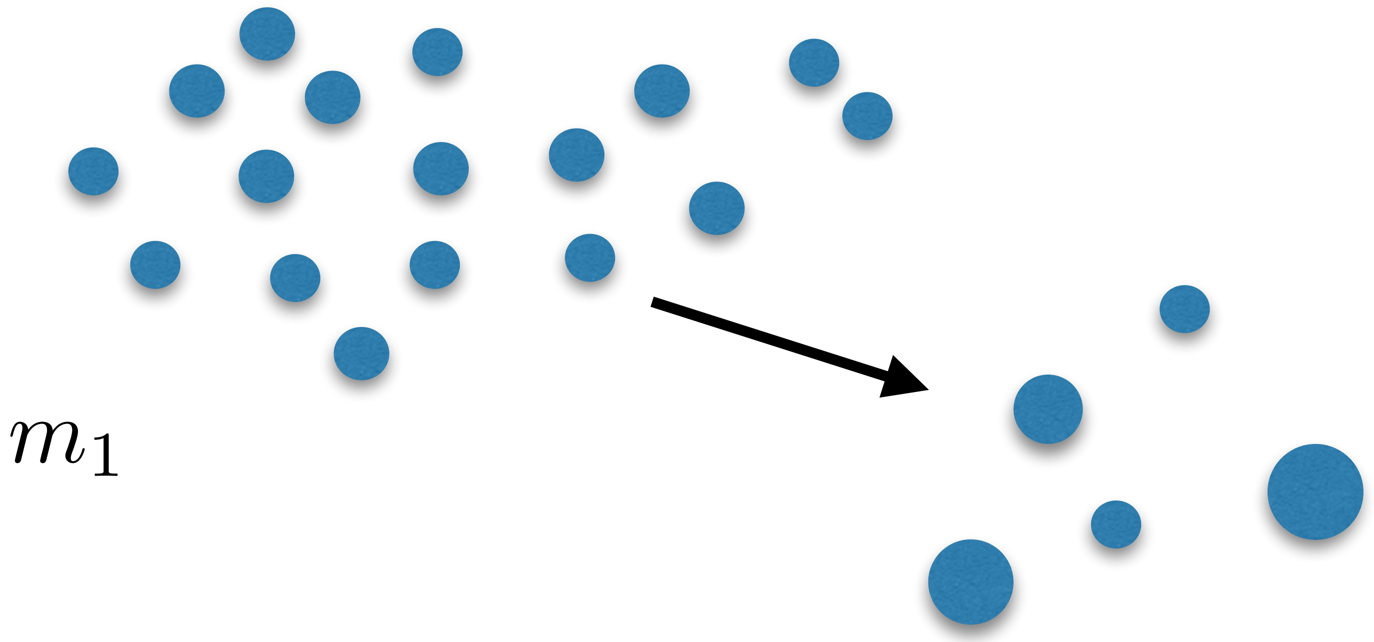
Richardson scaling

An application: Growth by coalescences

In both cases: very **dilute** solid particles suspended in a **turbulent** gas

Initially: almost mono-disperse size distribution

monomers with mass $\approx m_1$



How fast are large aggregates/drops created?

Time-evolution of the number $n_i(t)$ of particles with mass $i \times m_1$?

Kinetic approach

- **Smoluchowski** coagulation equation $i\,m_1 + j\,m_1 \xrightarrow{\kappa_{i,j}} (i+j)\,m_1$

$$\boxed{\dot{n}_i = \frac{1}{2}\sum_{j=1}^{i-1}\kappa_{i-j,j}\,n_{i-j}\,n_j - \sum_{j=1}^{\infty}\kappa_{i,j}\,n_i\,n_j}$$

$\kappa_{i,j}$: collision rate between particles with masses *i* and *j*

How is this global picture influenced by turbulent fluctuations?

- **Short-time asymptotics**

$n_1(t) \approx n_1(0)$ and creations are dominant

$$\dot{n}_2 = \frac{1}{2}\kappa_{1,1}n_1^2 \;\Rightarrow\; n_2(t) = \frac{1}{2}\kappa_{1,1}n_1^2\,t$$

$$\dot{n}_3 = \kappa_{1,2}\,n_1\,n_2 \;\Rightarrow\; n_3(t) = \frac{1}{4}\kappa_{1,1}\,\kappa_{1,2}\,n_1^3 t^2$$

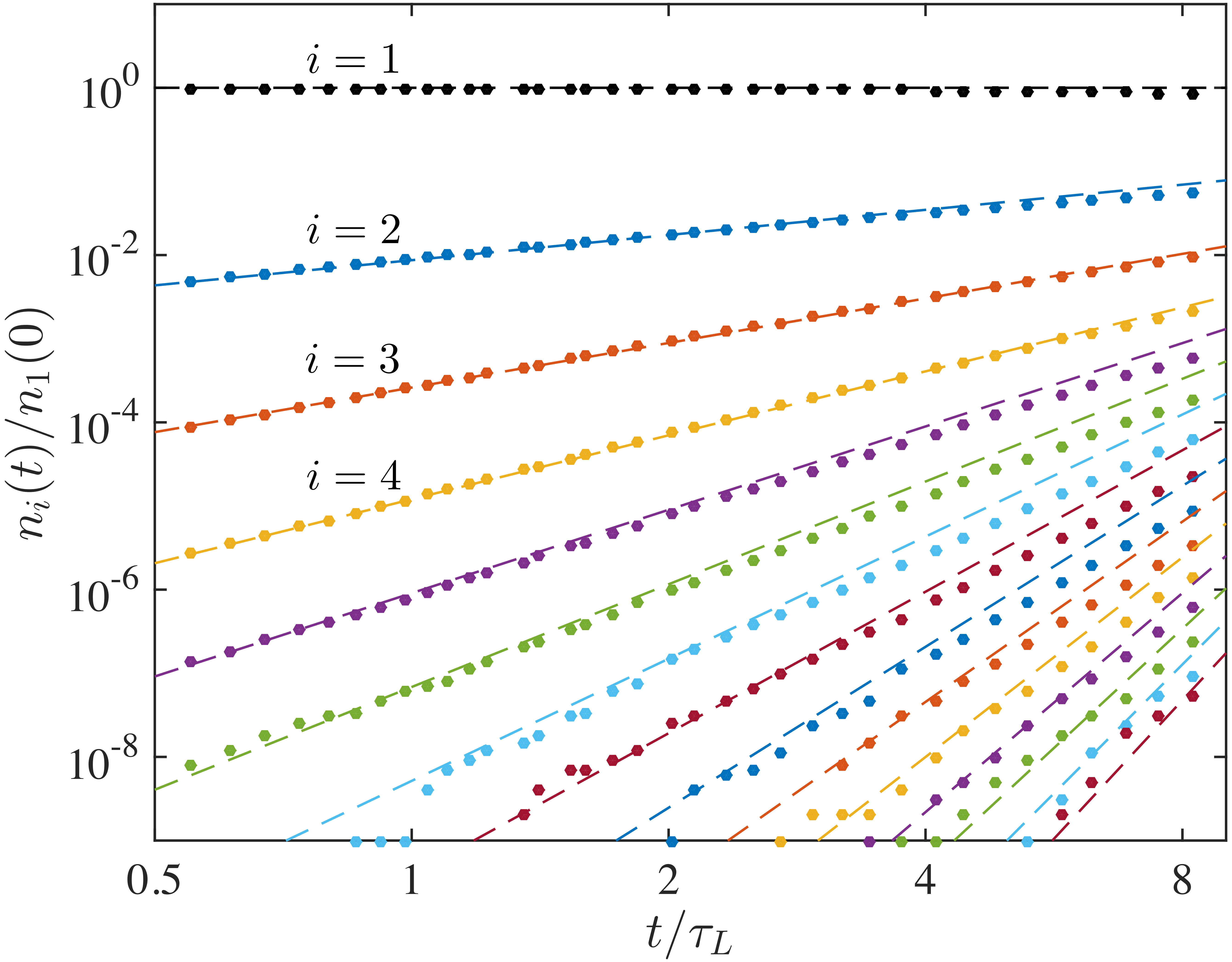
$$\boxed{n_i(t) \simeq n_1^i\left(\frac{t}{t_i}\right)^{i-1}}$$

The exponents do not depend on the kernel

Short-time growth of large particles

Numerics: incompressible Navier–Stokes

pseudo-spectral 2048^3 $(R_\lambda \approx 460)$ initially $n_1(0) = 10^9$ particles $a_1 \approx \eta/10$



Data show

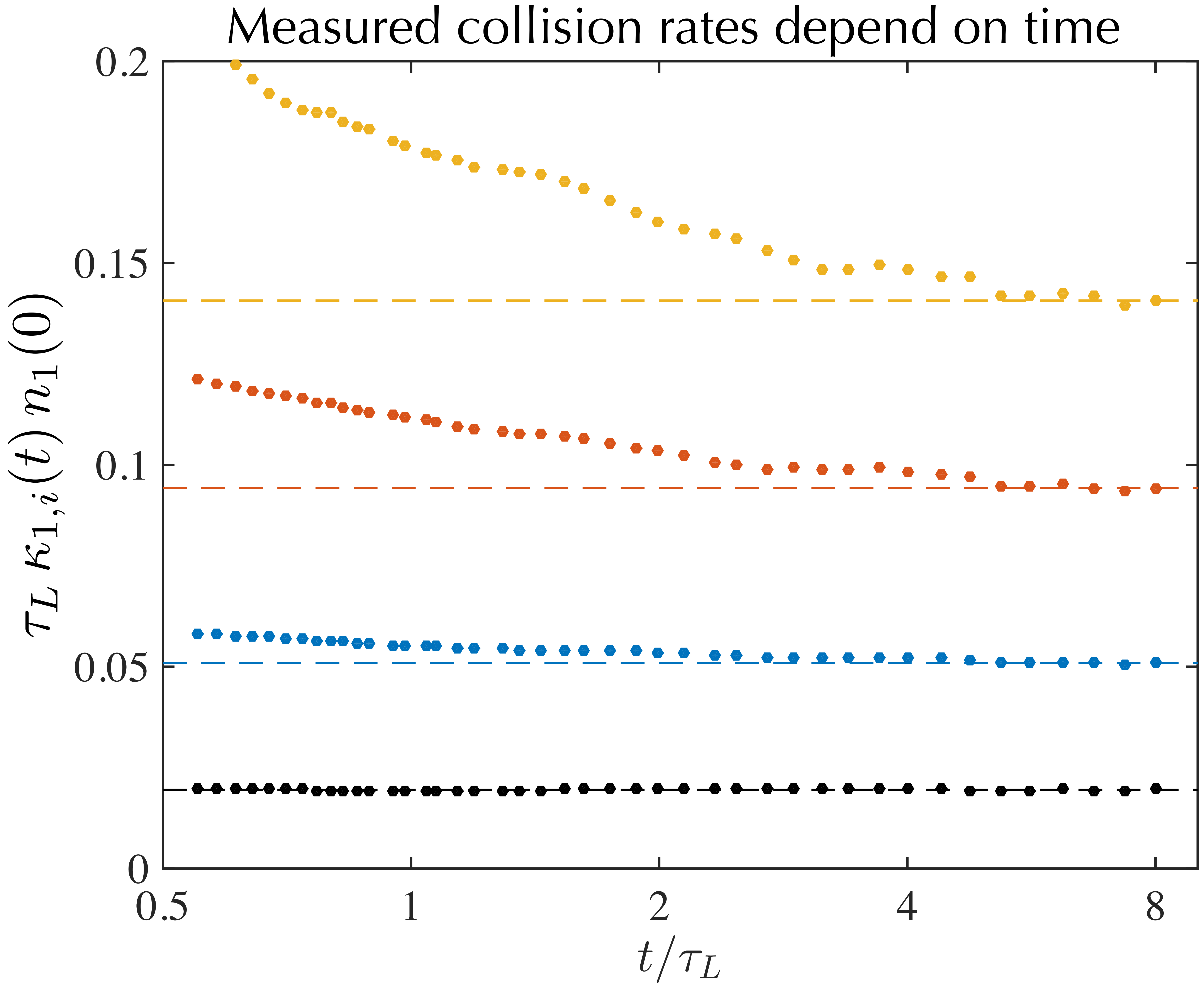
$n_i(t) \propto t^{0.73(i-2)+1}$

at short times

instead of $n_i(t) \propto t^{i-1}$

JB, Ray, Saw, Homann, *PRE* 2016

Kinetics not valid



Smoluchowski kinetics is not valid at short times / large sizes

Time evolution of the size distribution

- Back to basics:

Population balance

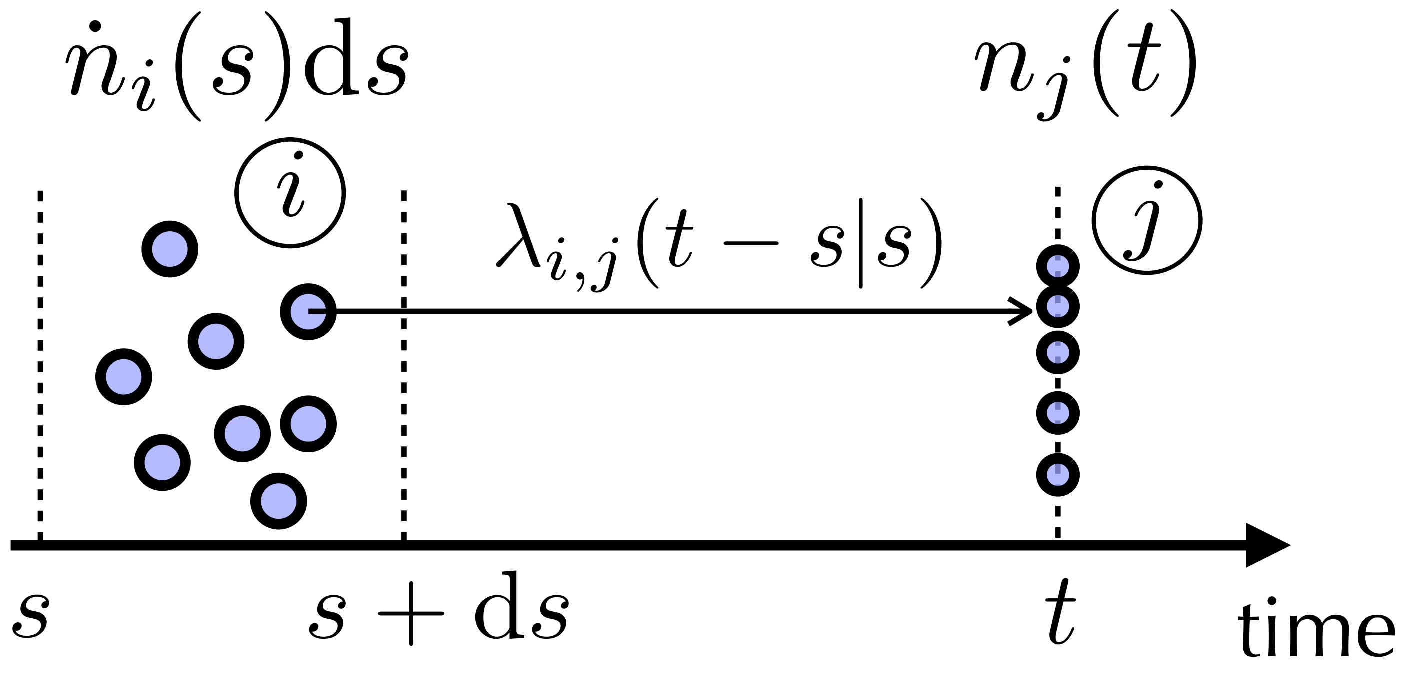
$$\dot{n}_i(t) = \frac{1}{2}\sum_{j=1}^{i-1} \mathcal{Q}_{i-j,j}(t) - \sum_{j=1}^{\infty} \mathcal{Q}_{i,j}(t)$$

$\mathcal{Q}_{i,j}(t)\,\mathrm{d}t$ average number of coalescences ⓘ + ⓙ in $[t, t+\mathrm{d}t]$

- Expression for the collision rate: $n_i(0) = 0$

$$\mathcal{Q}_{i,j}(t) = \int_0^t \lambda_{i,j}(t-s|s)\, n_j(t)\, \dot{n}_i(s)\, \mathrm{d}s$$

neglects possible correlations



- $\lambda_{i,j}(\tau|s)$ = rate at which a particle ⓘ, created at time s, coalesce with a particle ⓙ at time $s+\tau$

Inter-collision times

- The collisions define a **non-homogeneous Poisson process** with rate:

$$\lambda_{i,j}(\tau|s) = \lambda_{i,j}(\tau)$$

- Time to next collision: exponential distribution **with non-constant rate**

$$p_{i,j}(\tau) = \lambda_{i,j}(\tau)\,\mathrm{e}^{-\int_0^\tau \lambda_{i,j}(\tau')\,\mathrm{d}\tau'}$$

- **Smoluchowski kinetics:**

Successive coalescences are uncorrelated events

Memoryless process: $p_{i,j}(\tau)$ exponential $\Rightarrow$ $\lambda_{i,j}(\tau) = \mathrm{const} = \kappa_{i,j}$

coagulation kernels

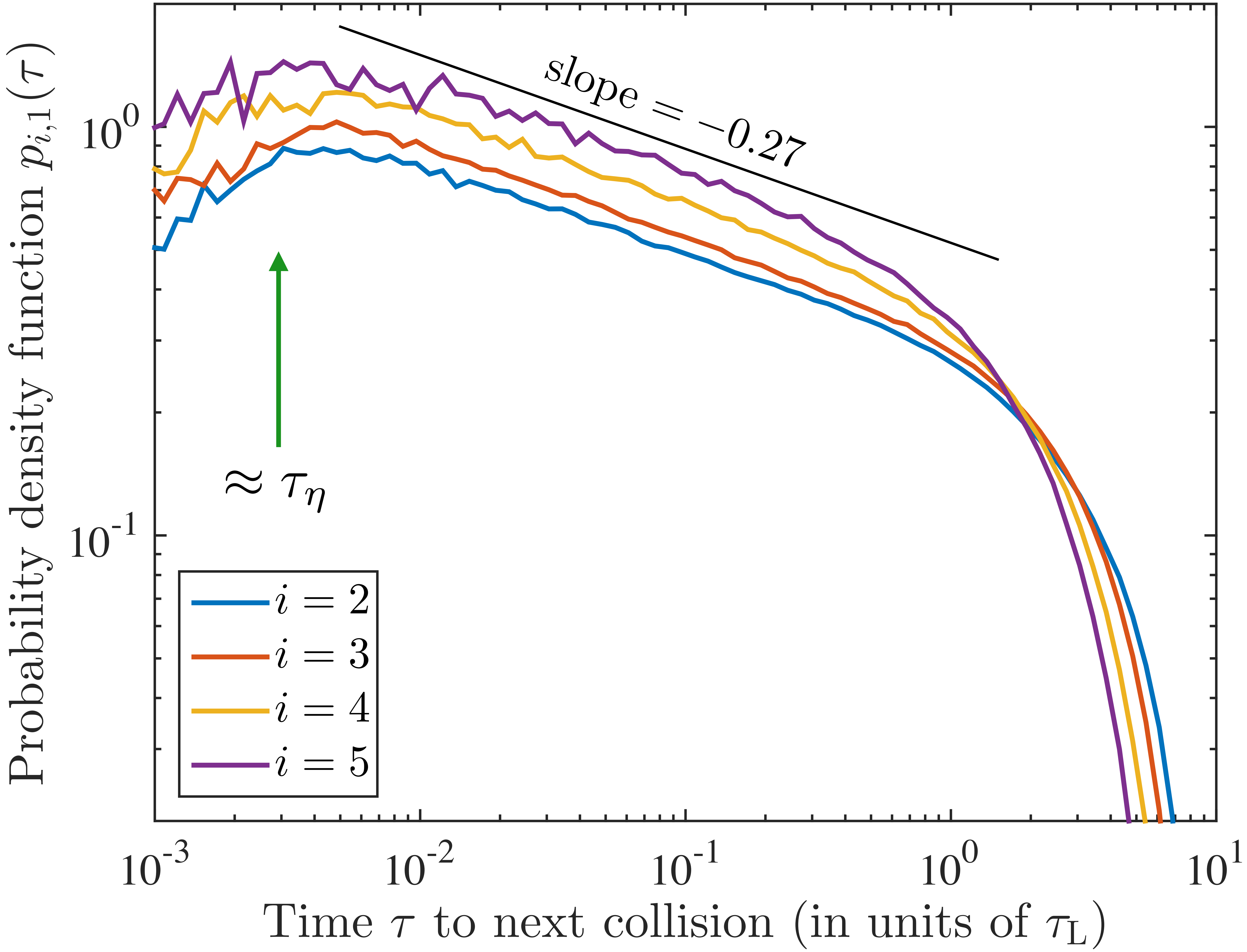
$$\mathcal{Q}_{i,j}(t) = \int_0^t \lambda_{i,j}(t-s) n_j(t)\,\dot{n}_i(s)\,\mathrm{d}s = \kappa_{i,j}\,n_i(t)\,n_j(t)$$

$$\dot{n}_i = \frac{1}{2}\sum_{j=1}^{i-1} \kappa_{i-j,j}\,n_{i-j}\,n_j - \sum_{j=1}^{\infty} \kappa_{i,j}\,n_i\,n_j$$

Long-range correlated collisions

Probability distribution of particles **mean-free times** (inter-collision times)

$$p_{i,j}(\tau) = \lambda_{i,j}(\tau)\,\mathrm{e}^{-\int_0^\tau \lambda_{i,j}(\tau')\,\mathrm{d}\tau'}$$



with $\lambda_{i,j} \propto \tau^{-0.27}$

Weibull distribution with shape parameter $k \approx 0.73$

Contribution from turbulent transport

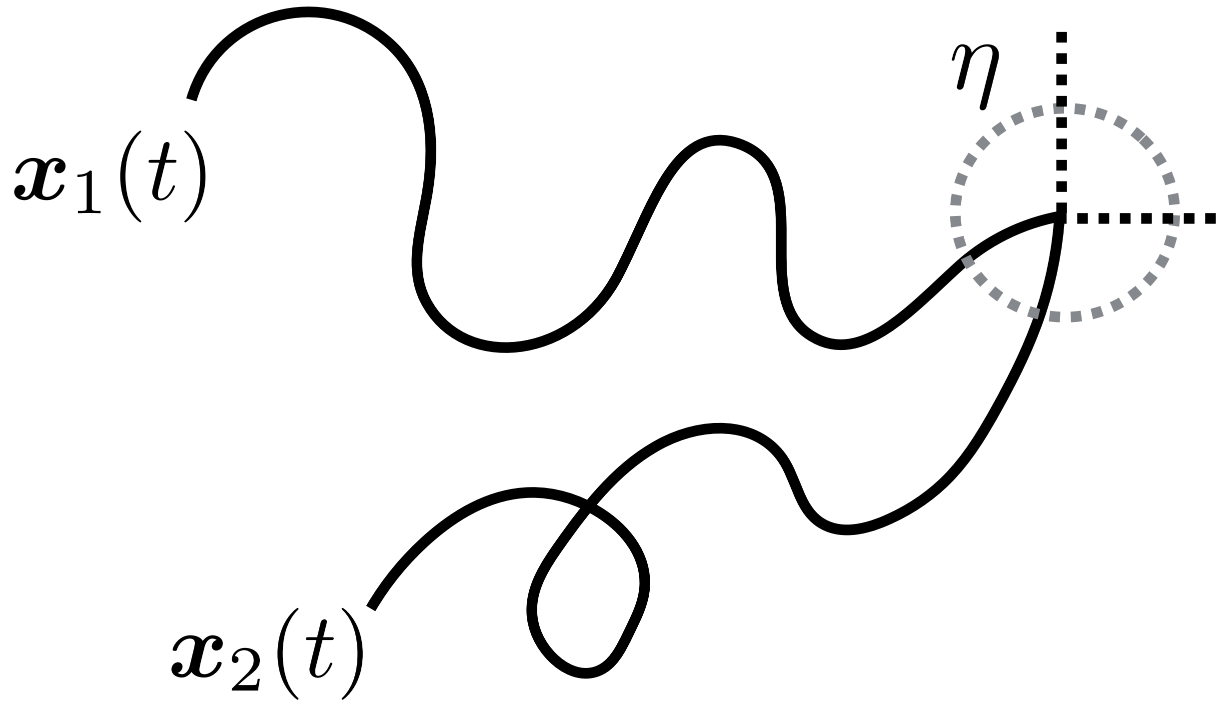
- **Dilute settings**: coalescing particles come from far apart

Two contributions to the coalescence rate:

$$\boxed{\lambda_{i,j}(\tau) = \lambda_{i,j}^{\mathrm{turb}}(\tau) \times \lambda_{i,j}^{\mathrm{micro}}(\tau)}$$

rate at which particles approach to a distance $\lesssim \eta$

probability they coalesce once they are at $r \lesssim \eta$



For $|\boldsymbol{x}_1 - \boldsymbol{x}_2| \gg \eta$ (inertial range)

Coalescing particles are almost tracers

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t) \qquad \boxed{|\boldsymbol{u}(\boldsymbol{x}_1) - \boldsymbol{u}(\boldsymbol{x}_2)| \sim |\boldsymbol{x}_1 - \boldsymbol{x}_2|^{1/3}}$$

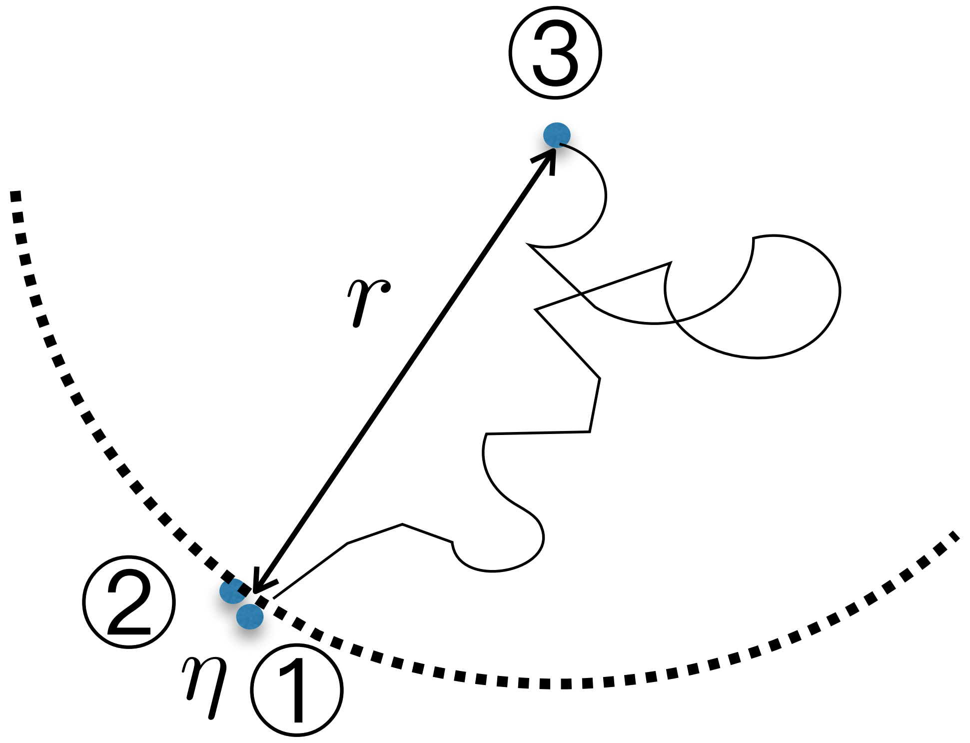
$|\boldsymbol{x}_1 - \boldsymbol{x}_2| \sim t^{3/2}$ (Richardson law)

For $|\boldsymbol{x}_1 - \boldsymbol{x}_2| \lesssim \eta$ details of the microphysics matters

finite size, inertia, hydrodynamical interactions, repulsive forces…

Dimensional estimates

Naive phenomenology:



Two contributions to the turbulent rate:

Density of particles ③ at distance r:

$$n(r) = r^2/L^3$$

Probability that a particle ③ initially at distance r approaches at a distance η from the newly created ①+②:

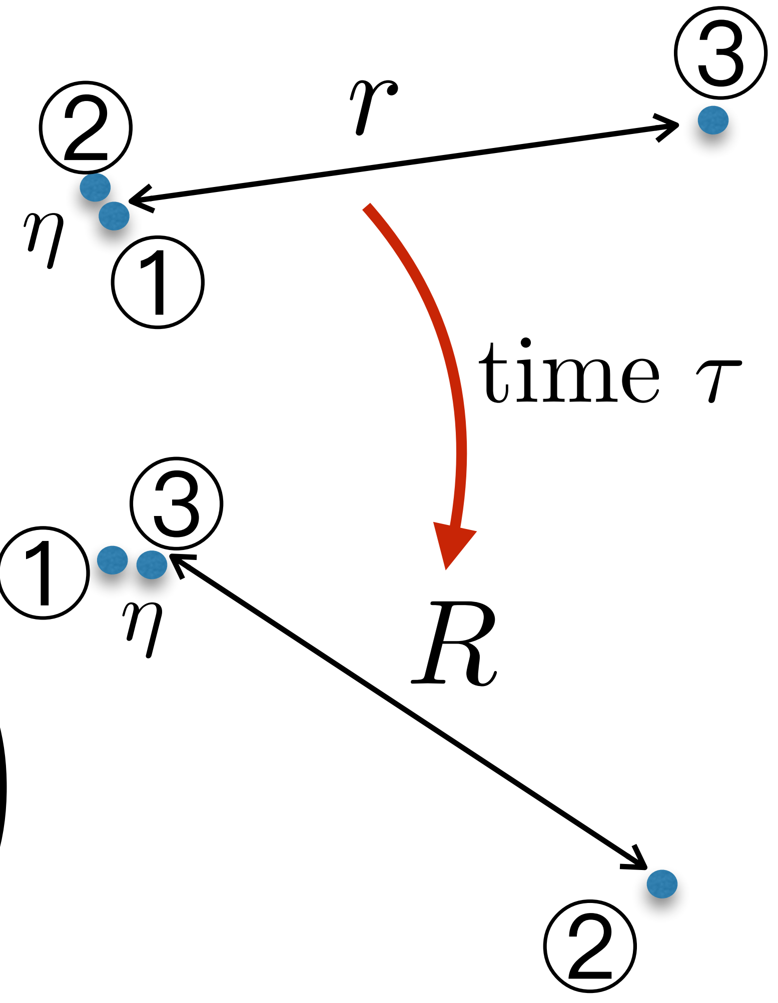
$$p(\eta,\tau\,|\,r,0) \simeq \underbrace{\left(\frac{\eta}{r}\right)^2}_{\text{solid angle}} \frac{1}{\tau^{3/2}} \underbrace{\Psi\left(\frac{r}{\tau^{3/2}}\right)}_{\text{Richardson scaling}}$$

Approaching rate:

$$\lambda_{i,j}^{\text{turb}}(\tau) \propto \int u_\eta\, p(\eta,\tau\,|\,r,0)\, n(r)\,\mathrm{d}r \sim \frac{\eta^2 u_\eta}{L^3} \int \Psi\left(\frac{r}{\tau^{3/2}}\right) \frac{\mathrm{d}r}{\tau^{3/2}} = \text{const}$$

Wrong! We are actually dealing with the 3-point motion

Actual turbulent rates



Collision rate: $\lambda_{i,j}^{\mathrm{turb}}(\tau) \simeq \int u_\eta \, p_3(\eta, \tau \,|\, r, R, 0) \, n(r) \, \mathrm{d}r \, \mathrm{d}R$

Again two contributions:

$$\begin{cases} n(r) = r^2/L^3 \\ \text{(unchanged)} \\ p_3(R, \eta, \tau | \eta, r, 0) = \left(\dfrac{\eta}{r}\right)^2 \left(\dfrac{r}{L}\right)^{\zeta_3 - 1} \dfrac{1}{\tau^3} \Phi\left(\dfrac{R}{\tau^{3/2}}, \dfrac{r}{\tau^{3/2}}\right) \end{cases}$$

(enhanced for small r)

$$\lambda_{i,j}^{\mathrm{turb}}(\tau) \propto \frac{1}{\tau^3} \int \left(\frac{r}{L}\right)^{\zeta_3 - 1} \Phi\left(\frac{R}{\tau^{3/2}}, \frac{r}{\tau^{3/2}}\right) \mathrm{d}r \, \mathrm{d}R \propto \left(\frac{\tau}{\tau_{\mathrm{L}}}\right)^{\frac{3}{2}(\zeta_3 - 1)} \qquad \tau \ll \tau_{\mathrm{L}}$$

Consequences on population dynamics:

$$\Rightarrow \quad \mathcal{Q}_{i,j}(t) \propto \int_0^t |t - s|^{\frac{3}{2}(\zeta_3 - 1)} n_j(t) \, n_i(s) \, \mathrm{d}s$$

$$\boxed{n_i(t) \propto t^{\left[1 - \frac{3}{2}(\zeta_3 - 1)\right](i-2)+1}}$$

$$\zeta_3 \approx 0.82 \quad \Rightarrow \quad n_i(t) \propto t^{0.73\,(i-2)+1}$$

Short-time growth is much faster than the kinetic prediction $\propto t^{i-1}$!

Conclusions

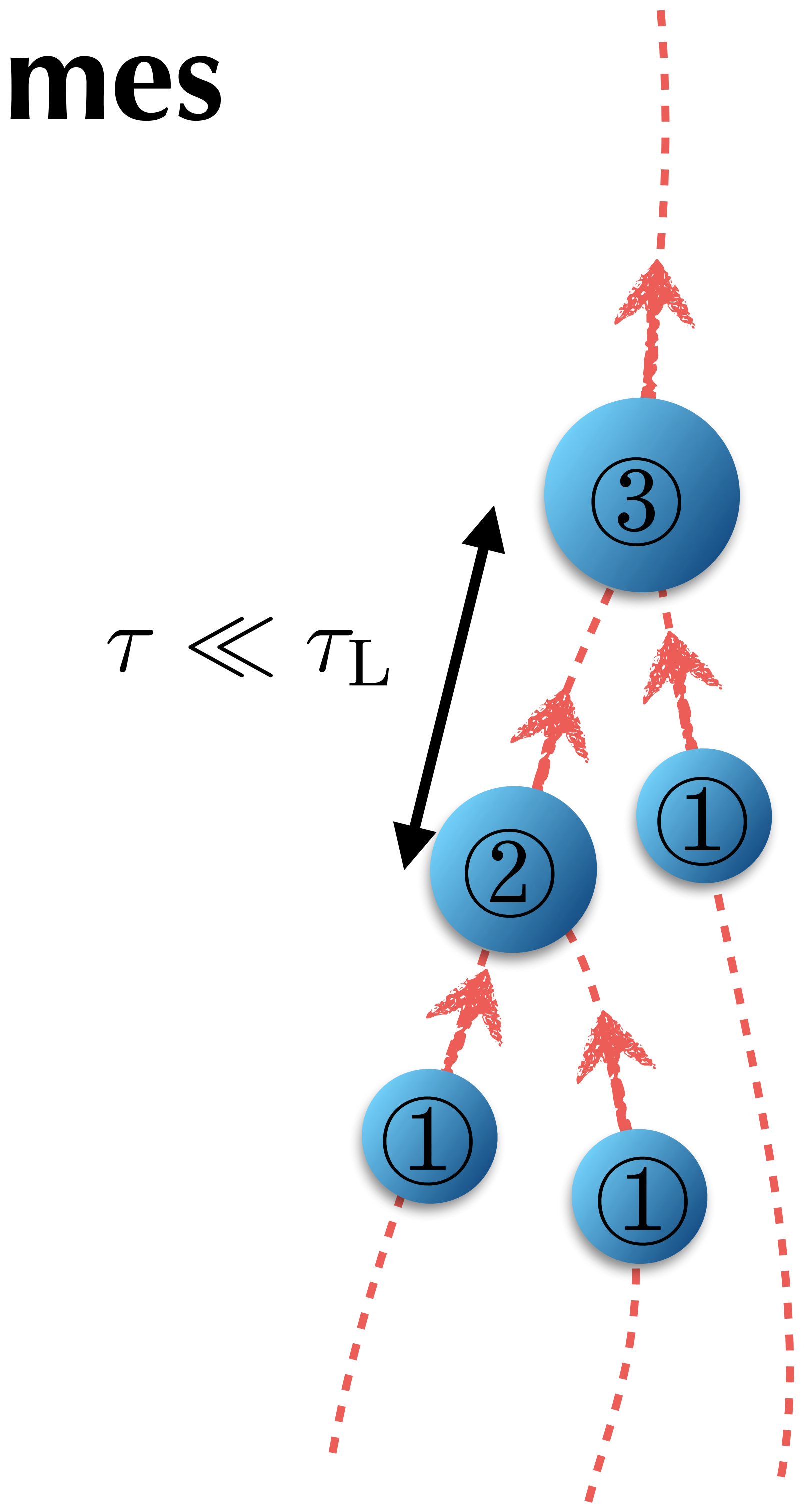
Kinetic approach for coagulation fails at short times

- Number of large particles grows as $n_i(t) \propto t^{0.73i}$ and not t^i

- "Rapid" successive collisions are correlated (mean-field breaks), when they involve inertial-range physics.

  This is a purely turbulent-mixing effect.

- New kinetic models (with e.g. multiple collisions) ?



Turbulent transport intermittency gives here the leading behavior