A Precipitating Quasi-Geostrophic Approximation

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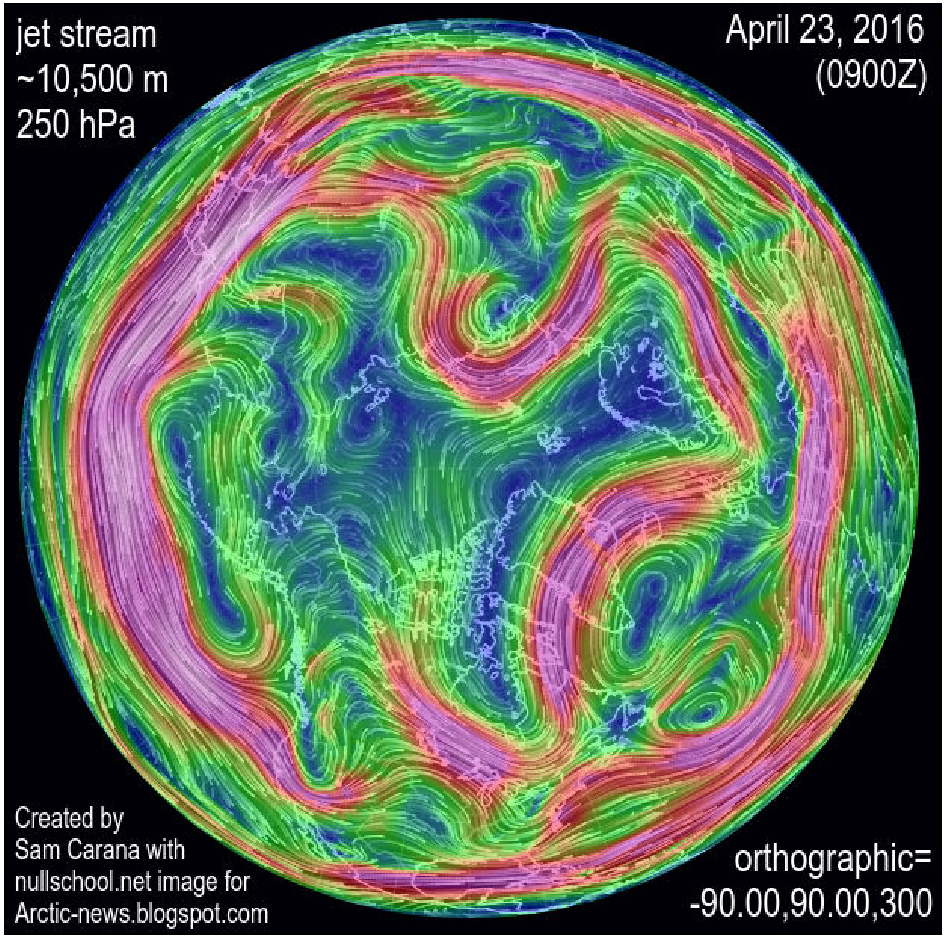

polar and subtropical jets merge over the Pacific

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Goals

Derive/analyze a precipitating QG model for large-scale mid-latitudes $==>$ effects of latent heat release for storms

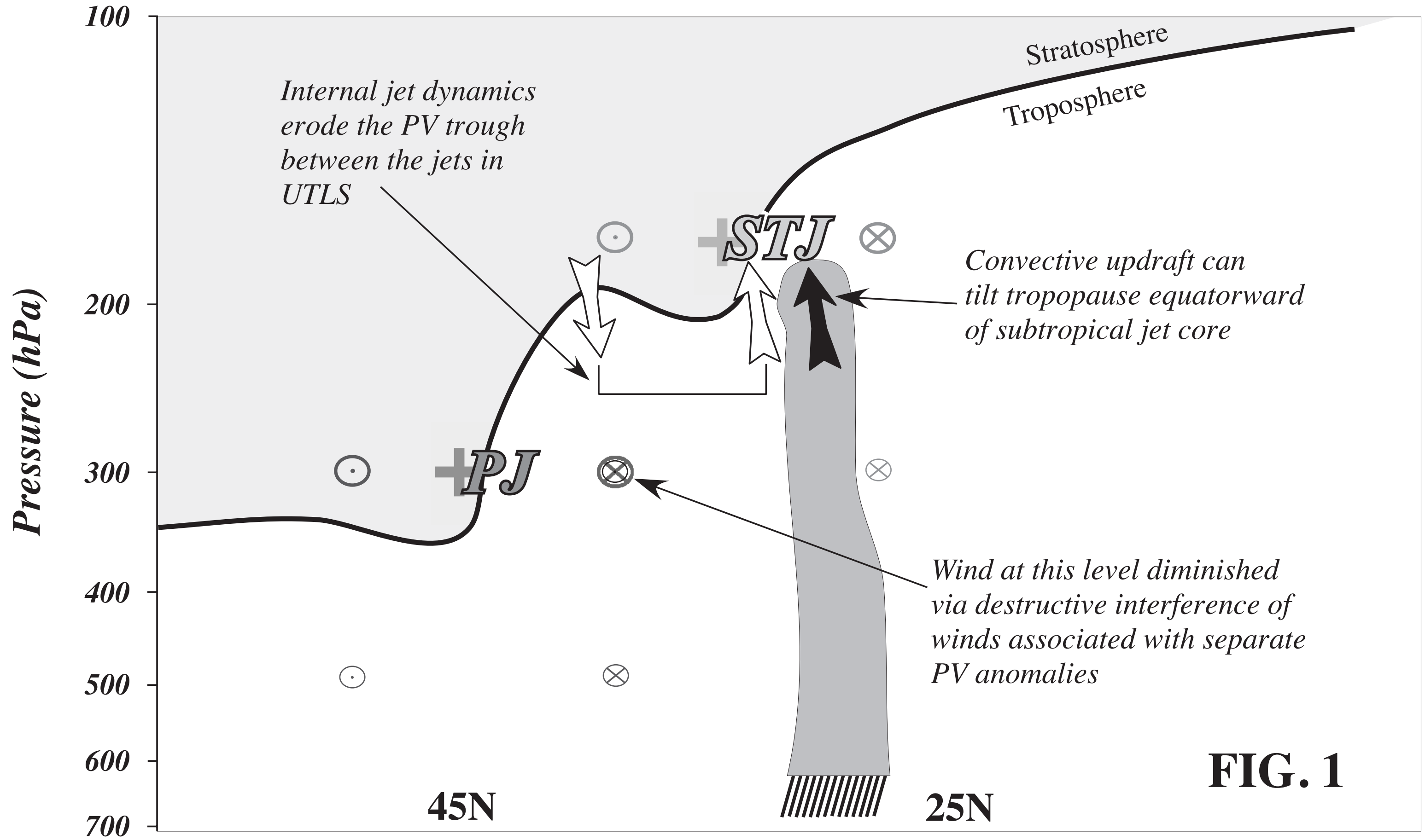


FIG. 1

Figure by Jon Martin/Andrew Winters

People




Outline

- Review the derivation of Dry QG
- Review Classic Eady Baroclinic Instability Analysis
- Derivation of PQG from FARE
  New features because of phase boundaries
- Warm Up: Eady with Water (no phase boundaries)
- Ongoing Work/Open Problems

QG Approximation:

Charney (1947) J. Meteor., The dynamics of long waves in a baroclinic westerly current.

Baroclinic Instability using QG:

Eady (1949) Tellus, Long waves and cyclone waves.

Baroclinicity proportional to $\nabla p \times \nabla \rho$ generates vorticity $\nabla \times \mathbf{u}$

The Dry Boussinesq Equations

$$\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\left(\frac{p}{\rho_0}\right) + \hat{\mathbf{z}}\, b$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{Db}{Dt} + N^2 w = 0, \quad b = \frac{g}{\theta_o}\theta, \quad N^2 = \frac{g}{\theta_o}\frac{d\tilde{\theta}}{dz}$$

Conservation of Potential Vorticity:

$$\frac{DPV}{Dt} = 0, \quad PV = (f + \nabla \times \mathbf{u}) \cdot \nabla(\tilde{\theta} + \theta)$$

Nondimensionalize

Specify characteristic scales: $U, L, H, \bar{P}, \Theta, W = UH/L$

$$\frac{D\mathbf{u}_h}{Dt} + Ro^{-1}\mathbf{u}_h^{\perp} + Eu\,\nabla_h p = 0$$

$$A^2\frac{Dw}{Dt} + Eu\,\frac{\partial p}{\partial z} - \Gamma\,A^2 b = 0$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\frac{Db}{Dt} + Fr^{-2}\,(\Gamma\,A^2)^{-1}\,w = 0$$

$$Ro = \frac{U}{fL},\quad Eu = \frac{\bar{P}}{\rho_o U^2},\quad Fr = \frac{U}{NH},\quad \Gamma = \frac{BH}{W^2} = g\frac{\Theta}{\theta_o}\frac{L^2}{U^2 H},\quad A = \frac{H}{L}$$

Choose a Distinguished Limit for $L \gtrsim L_d = NH/f$ deformation radius

$$Ro = Eu^{-1} = \epsilon, \quad Fr = \frac{L}{L_d} Ro = O(\epsilon), \quad \Gamma\, A^2 = Fr^{-1}$$

$$\frac{D\mathbf{u}_h}{Dt} + \epsilon^{-1}\mathbf{u}_h^{\perp} + \epsilon^{-1}\nabla_h p = 0$$

$$A^2 \frac{Dw}{Dt} - \epsilon^{-1}\frac{\partial p}{\partial z} - \epsilon^{-1}\frac{L_d}{L} b = 0$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{Db}{Dt} + \epsilon^{-1}\frac{L_d}{L}\, w = 0$$

$\theta_o \approx 300$ K, potential temperature anomaly scale $\Theta \approx 3$ K and mid-latitude horizontal velocity scale $U \approx 10 \text{ m s}^{-1}$ ==> $\epsilon \approx 0.1$

Expand

All variable $f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots$

Lowest order: Geostrophic, Hydrostatic Balance

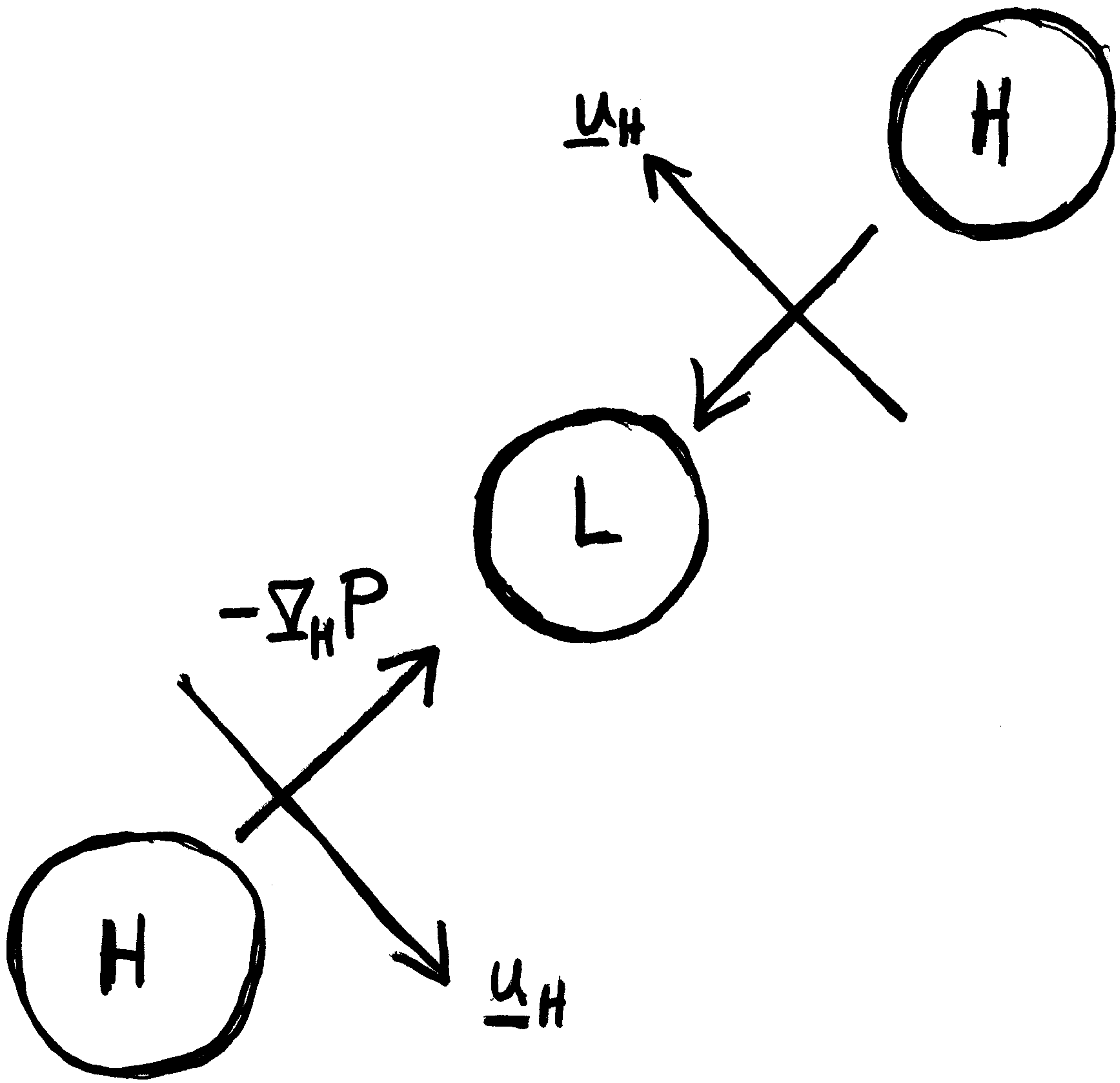
$$u^{(0)} = -\frac{\partial p^{(0)}}{\partial y}, \qquad v^{(0)} = \frac{\partial p^{(0)}}{\partial x}$$

$$\frac{\partial p^{(0)}}{\partial z} = \frac{L_d}{L} b^{(0)}$$

$$\nabla_h \cdot \mathbf{u}_h^{(0)} = 0, \qquad w^{(0)} = 0$$

Introduce a streamfunction $\psi = p^{(0)}$

Basic pressure gradient/velocity pattern of Geostrophic Balance



Next Order Corrections

Curl of Horizontal Momentum and Continuity:

$$\frac{D_h^{(0)}\zeta^{(0)}}{Dt} = \frac{\partial w^{(1)}}{\partial z}, \qquad \zeta^{(0)} = \nabla_h^2 \psi$$

Buoyancy:

$$\frac{D_h^{(0)} b^{(0)}}{Dt} + \frac{L_d}{L}\, w^{(1)} = 0, \qquad \frac{D_h^{(0)}}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Eliminate w:

$$\frac{D_h^{(0)}\zeta^{(0)}}{Dt} = -\frac{\partial}{\partial z}\left(\frac{L}{L_d}\frac{D_h^{0)} b^{(0)}}{Dt}\right)$$

Introduce Linearized PV

$$PV = \zeta^{(0)} + \frac{L}{L_d}\frac{\partial b^{(0)}}{\partial z}$$

Chain rule gives:

$$\frac{D_h^{(0)} PV}{Dt} = -\frac{L}{L_d}\hat{\mathbf{z}} \times \nabla_h \frac{\partial \psi}{\partial z} \cdot \nabla_h b^{(0)} = 0$$

where RHS=0 by ‘gradient wind balance.’

Elliptic Inversion

After each evolution step, need to find ψ by an inversion:

$$\frac{D_h^{(0)} PV}{Dt} = 0, \qquad \nabla_h^2 \psi + \frac{L^2}{L_d^2} \frac{\partial^2 \psi}{\partial z^2} = PV$$

Then compute velocities to be used in D_h/Dt at the next evolution step:

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$

$$\frac{D_h^{(0)}}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Eady Baroclinic Instability Analysis

$\psi = -\overline{U}zy + \psi'(x,y,t), \quad \mathbf{u} = -\partial\psi/\partial y$

(zonal flow with vertical shear)

$b^{(0)} = -(L/L_d)\overline{U}y + b'(x,y,z,t)$

(temperature decreasing from equator to pole)

Linearize and plug in:

$\psi' = \mathrm{Re}[\Psi(z)\exp(i(kx+ly-\omega t))]$

Dry Eady Results: $\overline{U} = 10 \text{ m s}^{-1}$, $d\tilde{\theta}/dz = 3 \text{ K km}^{-1}$

$$C = \frac{\omega}{k\overline{U}}, \qquad \mu_k^2 = (k^2 + l^2)\frac{L_d^2}{L^2}$$

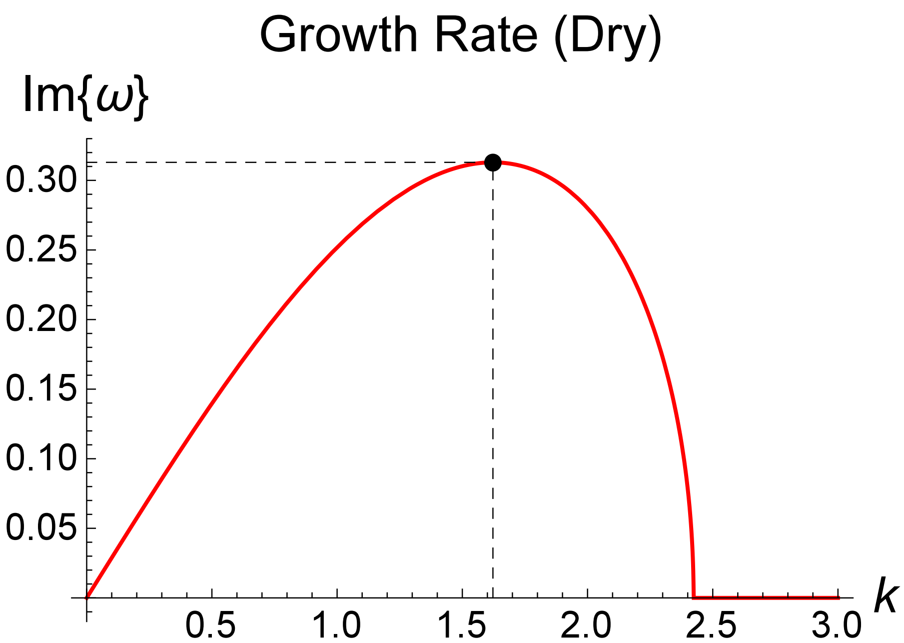
$$C_{\pm} = \frac{1}{2} \pm \frac{1}{2\mu_k}\left[(\mu_k - 2\tanh\frac{\mu_k}{2})(\mu_k - 2\coth\frac{\mu_k}{2})\right]^{1/2}$$

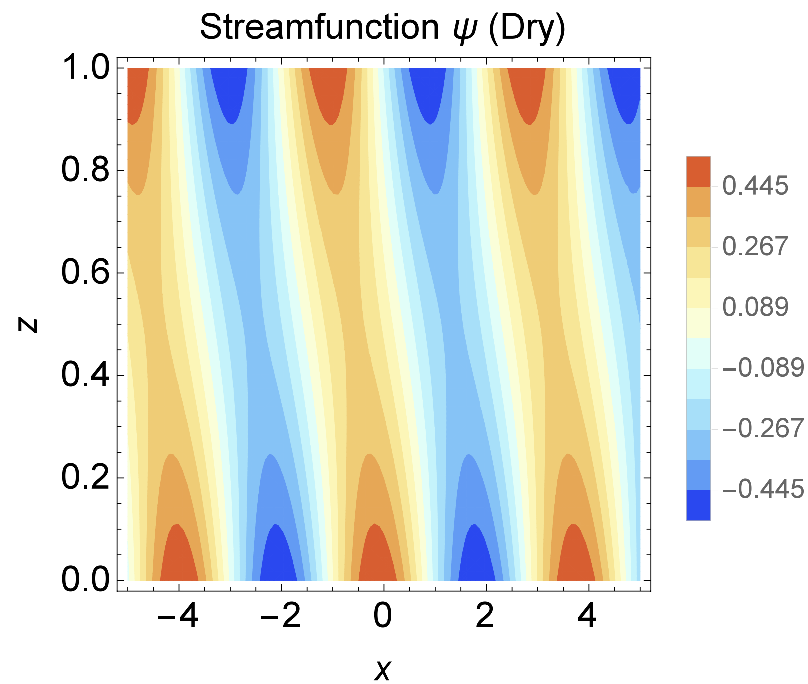
max σ = max $\mathrm{Im}(\omega)$ obtained for $l = 0$, $\mu_k \approx 1.6$,
$C_+(\mu_m) \approx 0.5 + 0.192898i$

$$\sigma \approx 0.31\frac{L}{L_d}\overline{U} \quad \text{(nondimensional)}$$

Growth rate $\sigma = 0.27 \text{ day}^{-1}$, Phase Speed Re$(\omega/k) = 5$ m/s
Wavelength $l_{\max} \approx 3,900$ km.

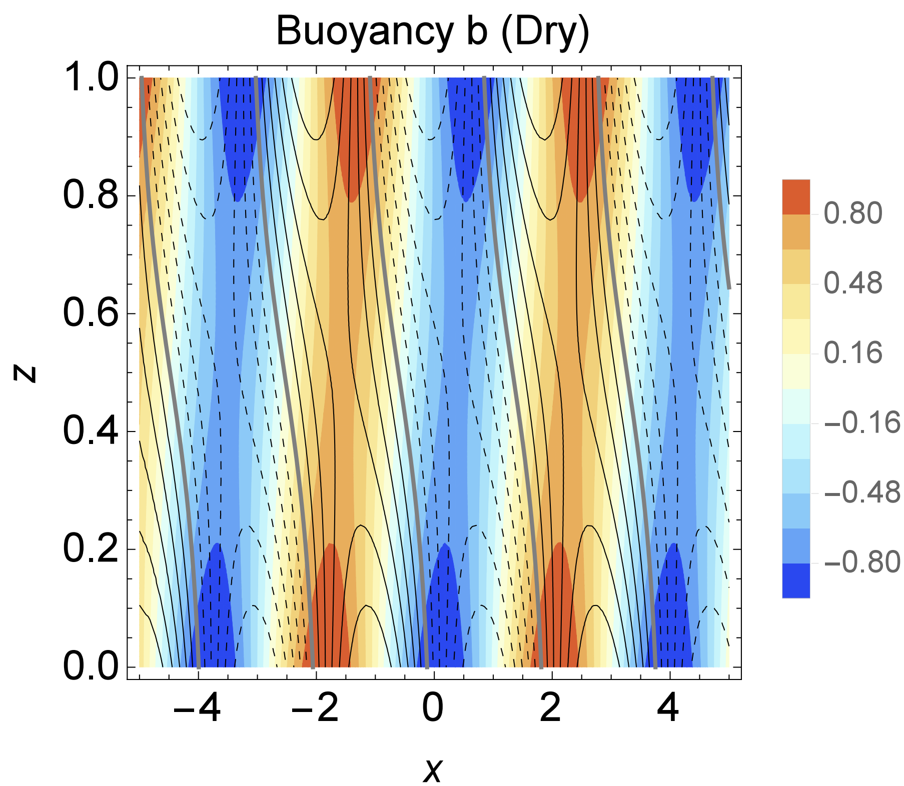
Dry Eady Results

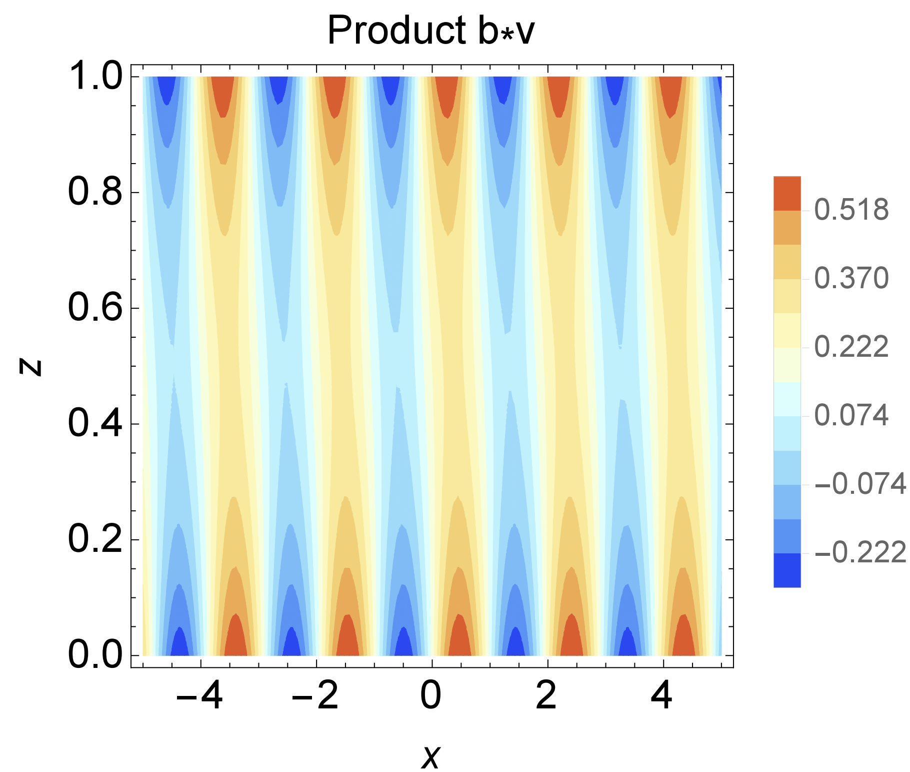




Left: Growth rate Im ω as a function of k for $l = 0$; Right: Dry streamfunction ψ for the most unstable mode.

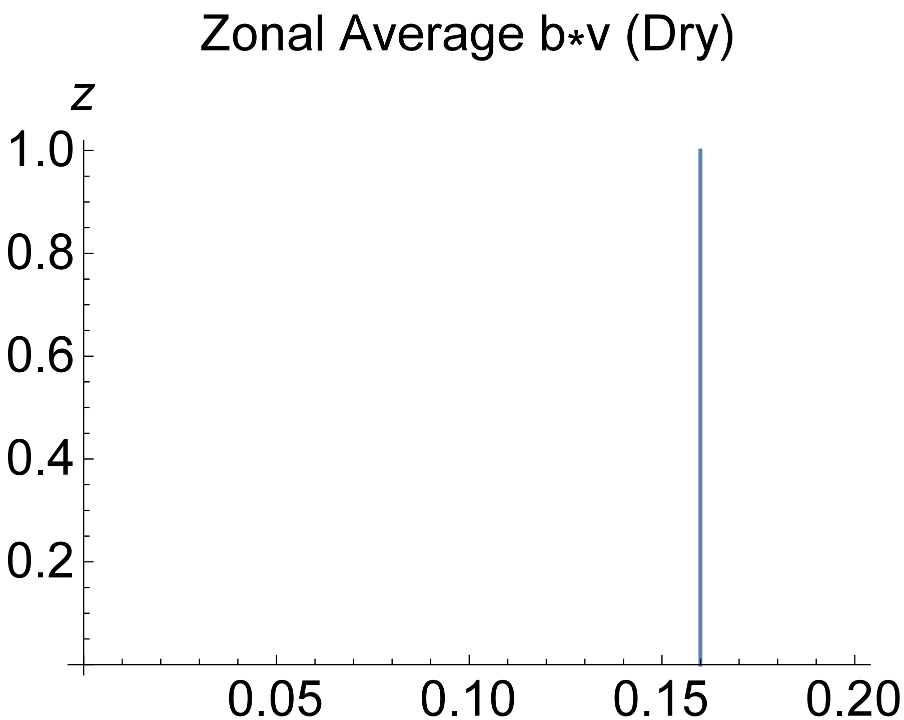
Meridional Flux of Buoyancy

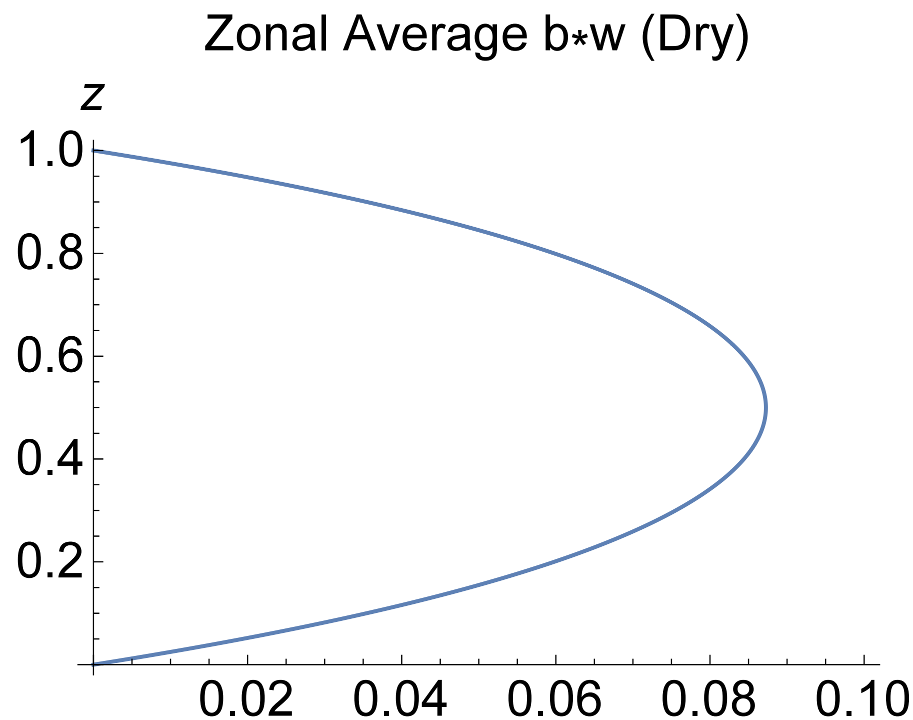




Left: Bouyancy with meridional velocity overlaid (solid lines are positive velocity);
Right: Meridional flux of buoyancy (at top and bottom, warm air (cold air) moves north (south).

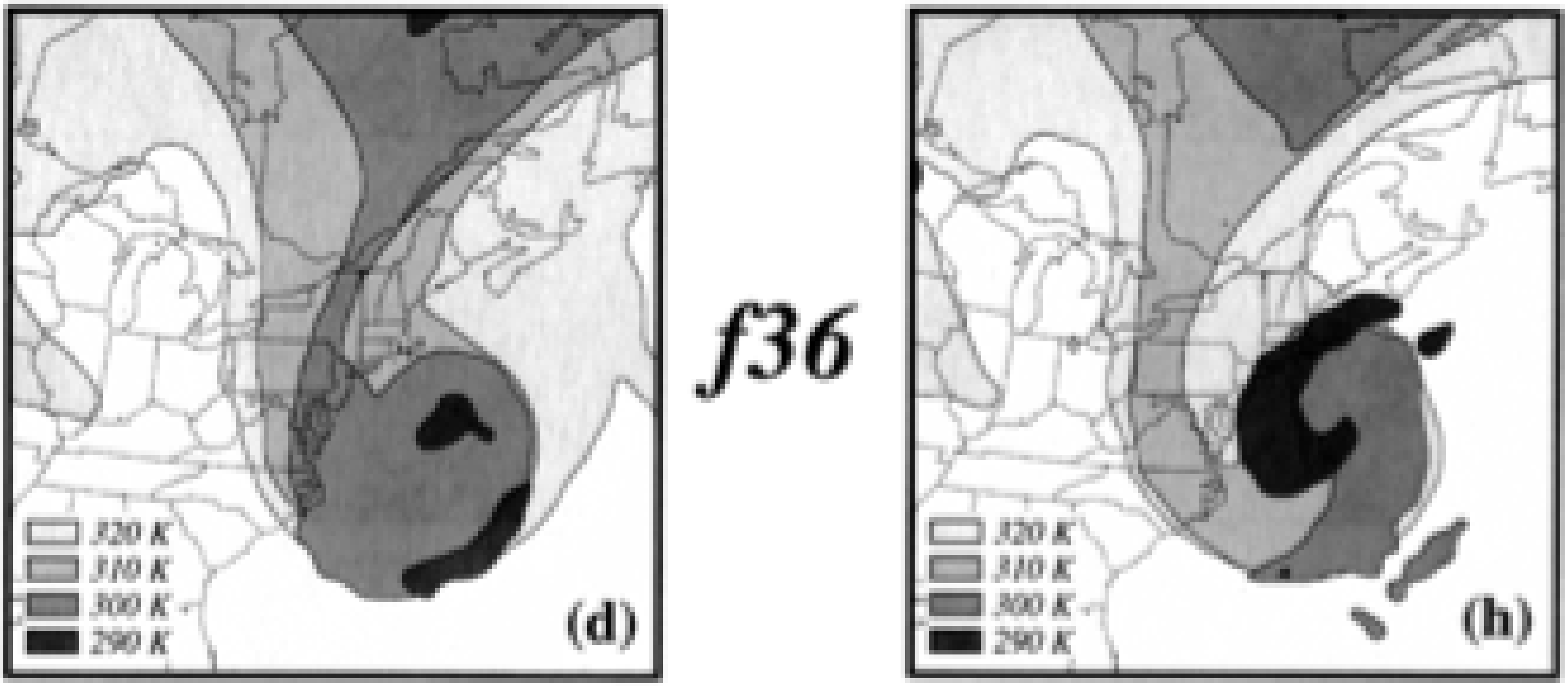
Zonally Averaged Fluxes





Left: Constant zonally averaged northward flux of heat; Right: Zonally averaged upward flux of heat; max at mid-altitudes

From Posselt & Martin 2004:



Mid-latitude Cyclone Development; 36-Hours; θ.
Left: NLHR Simulations; Right: FP Simulations

Derivation of a Precipitating QG:

$$\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\left(\frac{p}{\rho_0}\right) + \hat{\mathbf{z}}\,(b_u H_u + b_s H_s), \qquad \nabla \cdot \mathbf{u} = 0$$

$$\frac{Dq_t}{Dt} - V_T \frac{\partial q_r}{\partial z} = 0, \qquad \frac{D\theta_e}{Dt} = 0, \qquad \theta_e = \theta + (L_v q_v / c_p)$$

$$b_u = g\left[\frac{\theta_e}{\theta_o} + \left(R_{vd} - \frac{L_v}{c_p \theta_o}\right) q_t\right]$$

$$b_s = g\left[\frac{\theta_e}{\theta_o} + \left(R_{vd} - \frac{L_v}{c_p \theta_o} + 1\right) q_{vs}(z) - q_t\right]$$

$H_u = 1$, $\quad H_u = 0$ unsaturated, $\quad H_s = 1 - H_u$

Distinguished Limit; Non-dimensional Parameters

$$Ro = \epsilon, \qquad Fr = \frac{L}{L_d} Ro = O(\epsilon)$$

$L_d = NH/f$ is different for dry, unsaturated and saturated!

stable: q_t rapidly decreasing, θ_e rapidly increasing with z

$$-\frac{L_v}{c_p} \frac{d\tilde{q}_t/dz}{d\tilde{\theta}_e/dz} = O(1), \qquad V_r = O(1)$$

All variables are expanded:

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots$$

Lowest-Order Geostrophic, Hydrostatic Balance

$$u^{(0)} = -\frac{\partial \psi}{\partial y}, \quad v^{(0)} = \frac{\partial \psi}{\partial x}, \quad \zeta^{(0)} = \nabla_h^2 \psi$$

$$b_u^{(0)} H_u + b_s^{(0)} H_s = \frac{L}{L_{de}} \frac{\partial \psi}{\partial z}$$

where ψ is the pressure.

Next-Order Corrections

Curl of Horizontal Momentum and Continuity:

$$\frac{D_h \zeta^{(0)}}{Dt} = \frac{\partial w^{(1)}}{\partial z}$$

Equivalent Potential Temperature:

$$\frac{D_h \theta_e^{(0)}}{Dt} + \frac{L_{de}}{L} w^{(1)} = 0$$

Total Water:

$$\frac{D_h q_t^{(0)}}{Dt} - G_M \frac{L_{de}}{L} w^{(1)} = V_r \frac{\partial q_r^{(0)}}{\partial z}$$

Note extra equation and 3 appearances of vertical velocity

Equivalent System

$$\frac{D_h^{(0)}\zeta^{(0)}}{Dt} = \frac{\partial w^{(1)}}{\partial z}$$

$$\frac{D_h^{(0)}b_u^{(0)}}{Dt} + \frac{L_{du}}{L}\, w^{(1)} + V_r \frac{\partial q_r^{(0)}}{\partial z} = 0$$

$$\frac{D_h^{(0)}M^{(0)}}{Dt} - V_r \frac{\partial q_r^{(0)}}{\partial z} = 0, \quad \text{(no vertical velocity)}$$

Dimensional Definitions:

$$b_u = g\left[\frac{\theta_e}{\theta_o} + \left(R_{vd} - \frac{L_v}{c_p\theta_o}\right) q_t\right], \quad M = q_t + G_M\theta_e, \quad G_M = -\frac{d\tilde{q}_t/dz}{d\tilde{\theta}_e/dz}$$

Now Manipulate Vorticity and Buoyancy Equations (as for dry QG)

$$\frac{D_h^{(0)}\zeta^{(0)}}{Dt} = -\frac{\partial}{\partial z}\left(\frac{L}{L_{du}}\frac{D_h^{0)}b_u^{(0)}}{Dt} + V_r\frac{L}{L_{du}}\frac{\partial q_r^{(0)}}{\partial z}\right)$$

$$\frac{D_h^{(0)}M^{(0)}}{Dt} - V_r\frac{\partial q_r^{(0)}}{\partial z} = 0$$

Dimensional Definitions:

$$b_u = g\left[\frac{\theta_e}{\theta_o} + \left(R_{vd} - \frac{L_v}{c_p\theta_o}\right)q_t\right], \quad M = q_t + G_M\theta_e, \quad G_M = -\frac{d\tilde{q}_t/dz}{d\tilde{\theta}_e/dz}$$

PV_u formulation of PQG

$$PV_u \equiv \nabla_h^2 \psi + \frac{L}{L_{du}} \frac{\partial b_u^{(0)}}{\partial z}$$

$b_u^{(0)}$ defined in both unsaturated and saturated regions

$$\frac{D_h^{(0)} PV_u}{Dt} = -\frac{L}{L_{du}} \hat{\mathbf{z}} \times \nabla_h \frac{\partial \psi}{\partial z} \cdot \nabla_h b_u^{(0)} - \frac{\partial}{\partial z} \left(V_r \frac{L}{L_{du}} \frac{\partial q_r^{(0)}}{\partial z} \right)$$

$$\frac{D_h^{(0)} M^{(0)}}{Dt} - V_r \frac{\partial q_r^{(0)}}{\partial z} = 0, \qquad M \equiv q_t + G_M \theta_e$$

Need jump conditions at phase boundaries

Iterative PV_u inversion to find ψ, H_u, H_s

A nonlinear elliptic equation:

$$\nabla_h^2\psi + \frac{L^2}{L_{du}^2}\frac{\partial}{\partial z}\left[H_u\frac{\partial\psi}{\partial z}\right] +$$

$$+D_M\frac{L^2}{L_{du}^2}\frac{\partial}{\partial z}\left[H_s\left(\frac{\partial\psi}{\partial z} + \frac{L_{du}}{L}(M - D_M q_{vs})\right)\right] = PV_u.$$

Old known locations of phase boundaries serve as the initial guess of iteration; $D_M = 1 + G_M > 1$ constant; $M = q_t + G_M\theta_e$.

Moist Baroclinic Instability: Related Literature

Emanuel, Fantini, Thorpe: 1987; 2-layer, semi-geostrophic; ascending (descending) saturated (dry) air; baroclinic instability

Lambaerts, Lapeyre, & Zeitlin 2012 Moist baroclinic instability in a simplified two-layer atmospheric model with condensation and latent heat release

Lapeyre, Held: 2004; 2-layer QG, turbulence study; parameterization of precip/latent heat release; jets (cyclonic vortices) dominate for weak (strong) latent heat release

Booth, Polvani, O'Gorman, Wang: 2015; extend dry theories using effective static stability; moisture $==>$ reduced stability parameterized by area fraction of upward motion

Eady Baroclinic Instability (either phase; away from phase boundaries)

$$\psi = -\overline{U}zy + \psi'(x, y, t)$$

(zonal flow with vertical shear)

$$b^{(0)} = -(L/L_d)\overline{U}y + b'(x, y, z, t)$$

(temperature decreasing from equator to pole)

New feature (optional)

$$q_t^{(0)} = -\overline{Q}y + q_t'(x, y, x, t)$$

(water decreasing from equator to pole)

$$\psi' = \mathrm{Re}[\Psi(z)\exp(i(kx+ly-\omega t))], \quad q_t' = \mathrm{Re}[Q(z)\exp(i(kx+ly-\omega t))]$$

The Moist Dynamics is the Same!

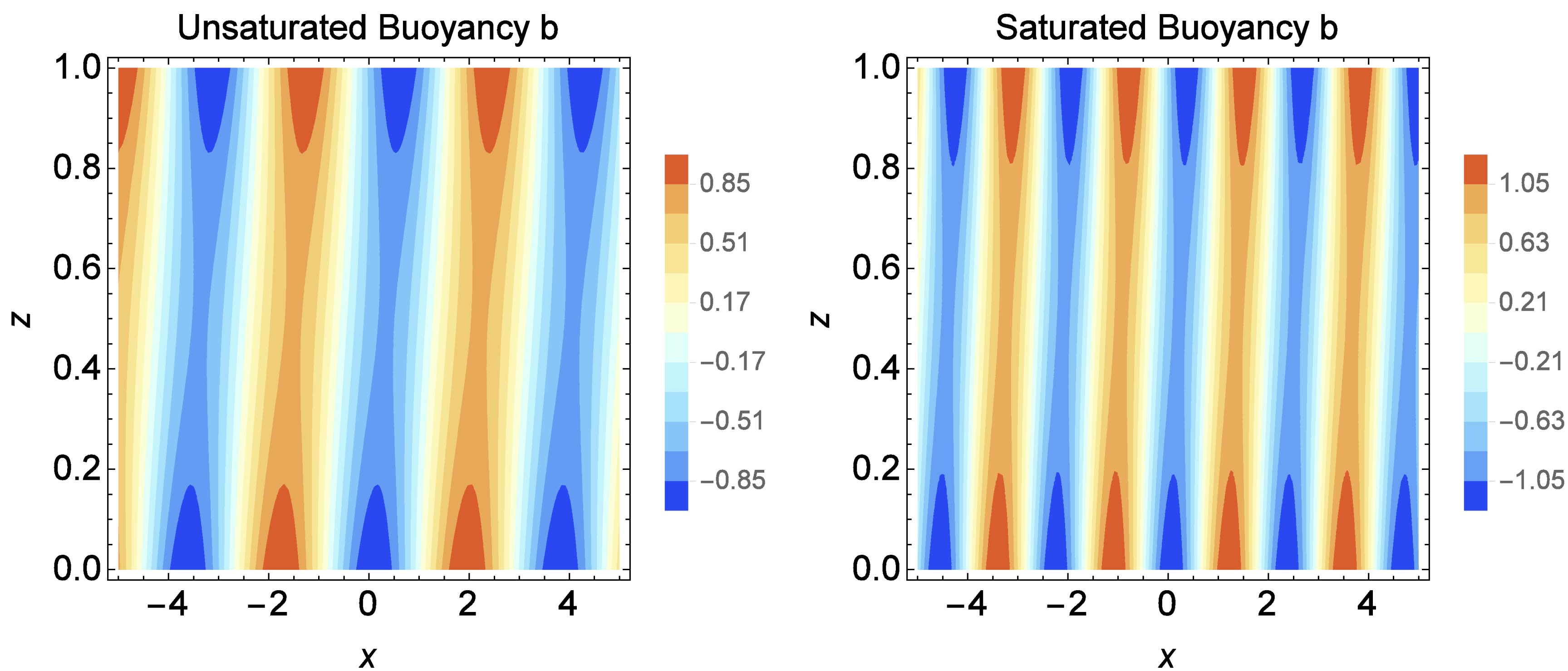
$$\frac{D_h^{(0)} PV}{Dt} = 0, \quad PV = \nabla_h^2 \psi + \frac{L}{L_d} \frac{\partial b^{(0)}}{\partial z}, \quad b^{(0)} = \frac{L}{L_d} \frac{\partial \psi}{\partial z}, \quad 0 < z < 1$$

$$\frac{D_h b^{(0)}}{Dt} = 0, \quad z = 0, 1 \quad (w = 0)$$

Simply a different L_d!

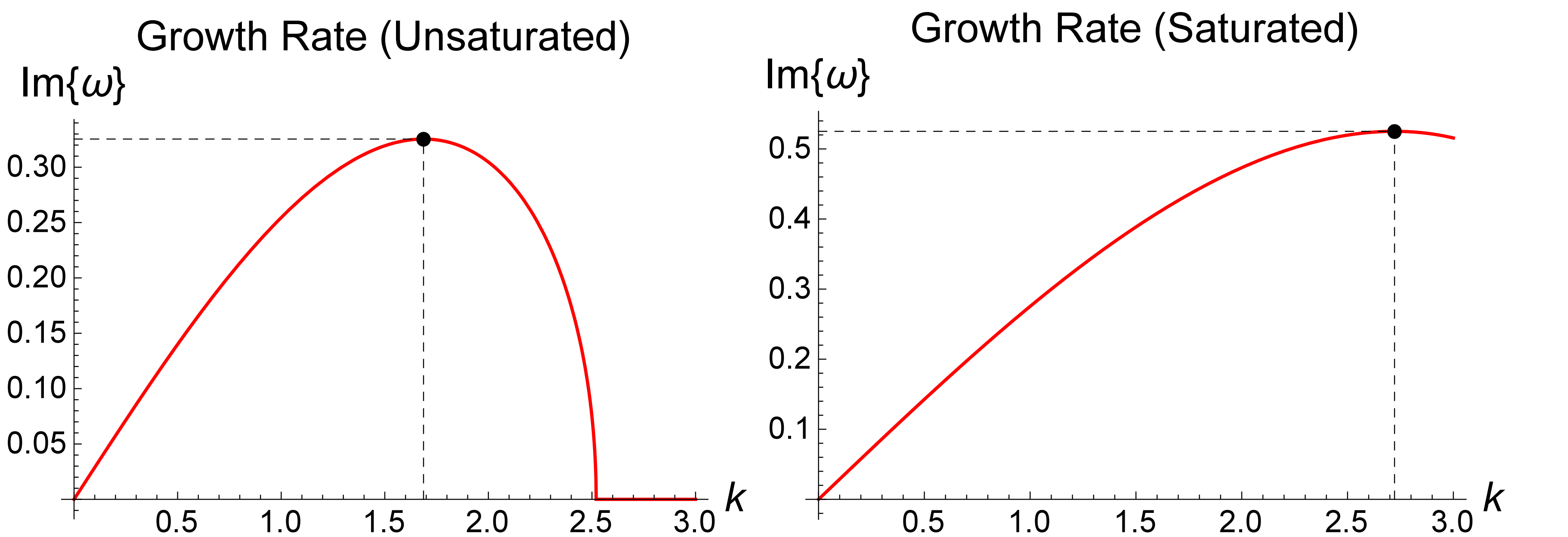
A Mathematical Equivalence; justifies 'dry PV thinking,' e.g. effective static stability

Comparison of Eady Anomalies; Buoyancy



Most unstable mode of non-dimensional buoyancy anomaly (normalized by energy); Left (Right): Unsaturated (Saturated) buoyancy anomaly. Saturated wavelength smaller: L_{ds}/L_{du}.

Growth Rate vs Wavenumber Comparison



Saturated Case: Growth rate increases by 61% ($L_{ds}/L_{du} = 0.61$);
Zonally averaged meridional flux of buoyancy increases by 61%;
Max zonally averaged vertical flux of buoyancy quadrupled.

(Emanuel, Fantini, Thorpe 1987)

Unsaturated Case: Determination of Water Vapor

$$\frac{D_h^{(0)} M^{(0)}}{Dt} = 0, \qquad M^{(0)} = q_v^{(0)} + G_M \theta_e^{(0)}$$

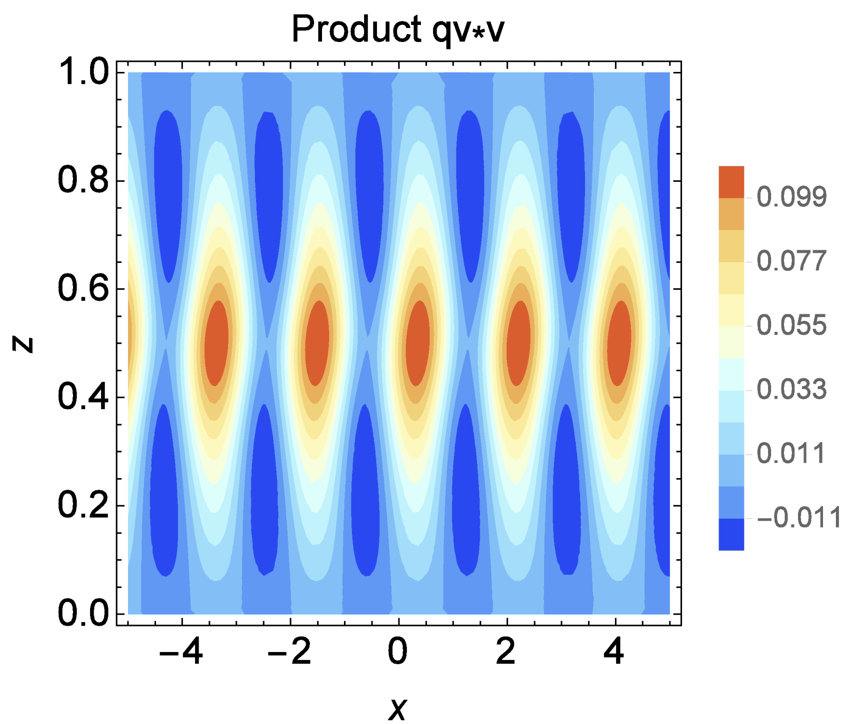
(no z-derivatives)

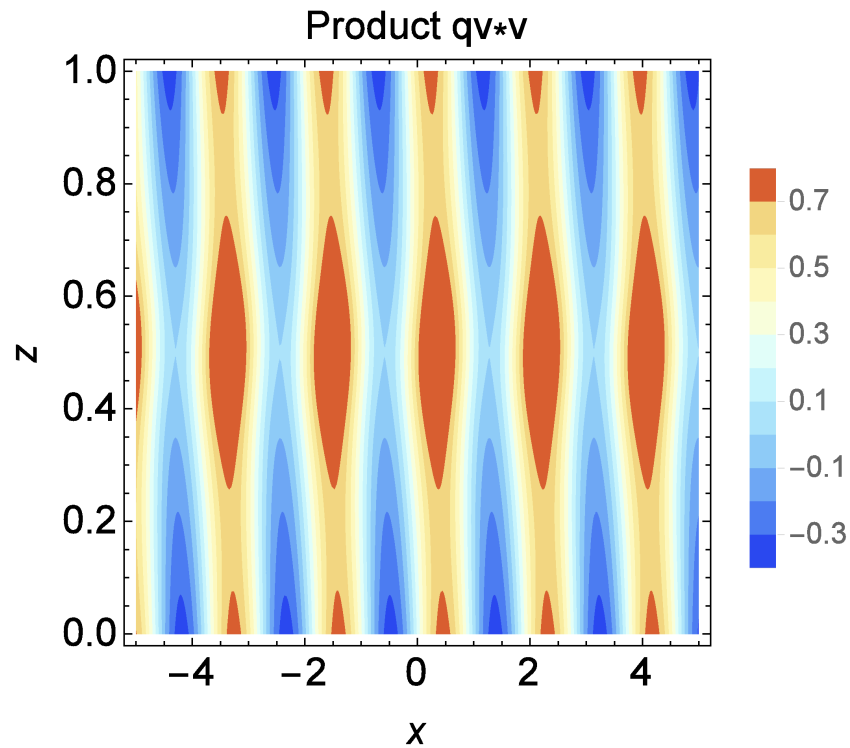
$$q_t^{(0)} = -\overline{Q} y + q_v', \qquad q_v' = \mathrm{Re}\Big(Q_v(z) \exp[i(kx + ly - \omega t)] \Big)$$

$$Q_v = -\frac{G_M}{1 + G_M} \frac{L}{L_{du}} \left(\frac{d\Psi}{dz} + \frac{1}{(C - z)} \Psi \right) - \frac{\overline{Q}_v}{\overline{U}} \frac{\Psi}{C - z} \quad \text{for } 0 < z < 1$$

An algebraic relation!

New! Fluxes of Water Vapor (Unsaturated)





Meridional flux of water vapor anomaly; Left: $\overline{Q}_v = 0$; Right: $\overline{Q}_v \neq 0$ (Eady background decreasing from equator to pole)

Saturated Case: Determination of Rain Water

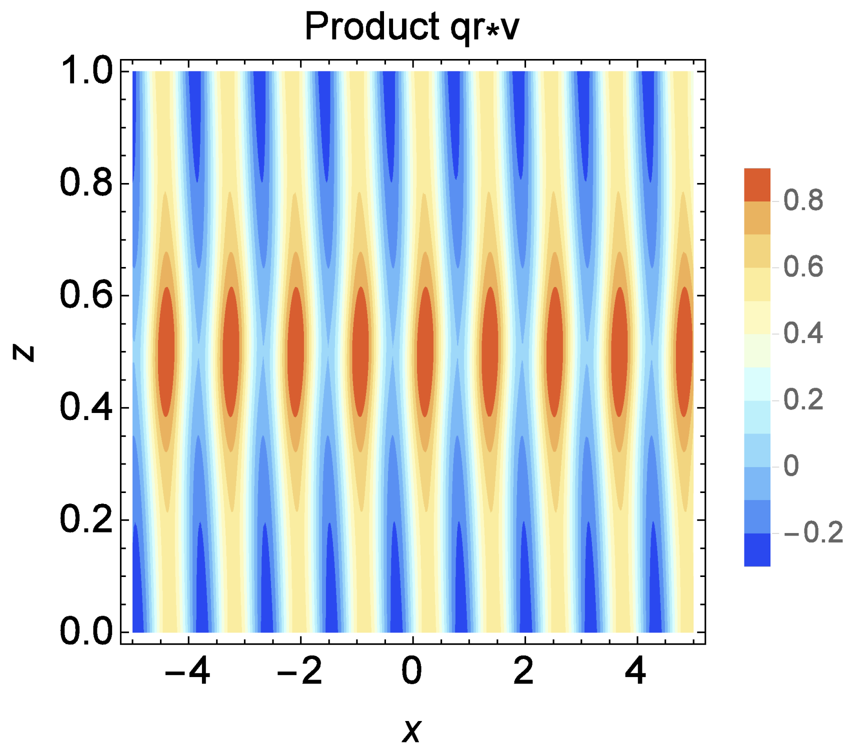
$$\frac{D_h^{(0)} M^{(0)}}{Dt} = V_r \frac{\partial q_r^{(0)}}{\partial z}, \qquad M^{(0)} = q_r^{(0)} + G_M \theta_e^{(0)}$$

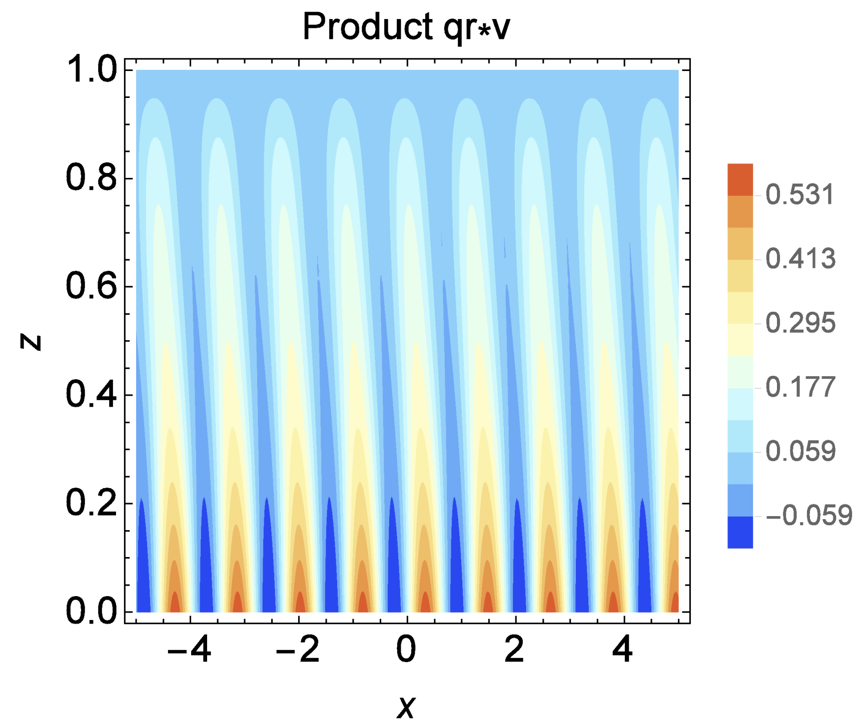
$$q_r^{(0)} = -\overline{Q}_r y + q_r', \qquad q_r' = \mathrm{Re}\Big(Q_r(z) \exp[i(kx + ly - \omega t)] \Big)$$

$$\frac{dQ_r}{dz} + i\frac{k\overline{U}}{V_r}(C - z)\,Q_r + i\frac{k\overline{U}}{V_r} G_M \frac{L}{L_s}\left((C - z)\frac{d\Psi}{dz} + \Psi \right) + i\frac{k\overline{U}}{V_r}\frac{\overline{Q}_r}{\overline{U}}\Psi = 0$$

1st order, linear ODE with $Q_r(1) = 0$ at top

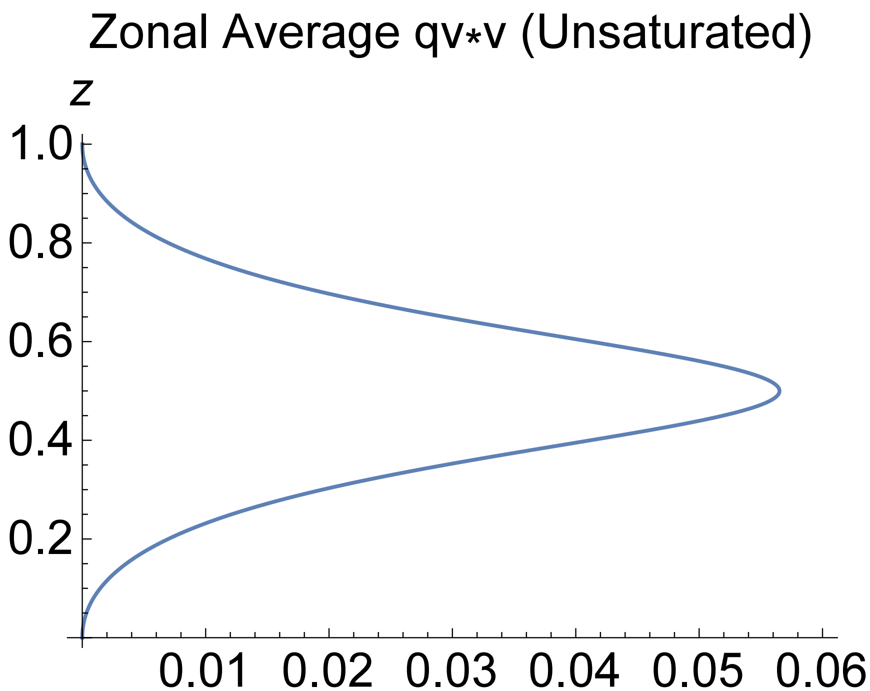
New! Fluxes of Liquid Water (Saturated)

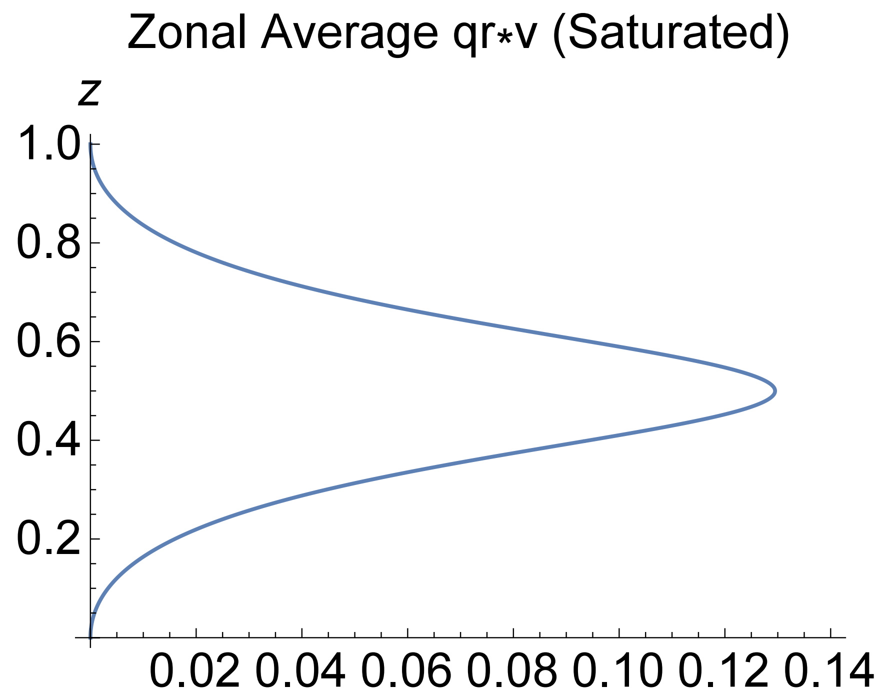




Meridional flux of rain water anomaly; Background rain decreasing with latitude; Left: $V_T = 0$; Right: $V_T = O(1)$.

Comparison Unsaturated vs. Saturated





Zonally averaged meridional flux of water anomaly; No background water profile; Left: Unsaturated; Right: Saturated with $V_T = 0$

Summary:

- More systematic derivation of moist QG

- New features: Additional equation, Loss of gradient wind balance across phase boundaries, nonlinear elliptic inversion equation

- Trigger for Precipitation is q_t; Form of $C_d - E_r \propto w$

- Linearized: Mathematical equivalence of Dry/Unsat/Sat dynamics; Water is a passive scalar

- Allows for including vertical structure of vertical & meridional fluxes of water in baroclinic instability

Current/Future Work

• Simple exact solutions with a phase change,

e.g. saturated cold core transitioning to unsaturated outer core; axisymmetric

• Numerical methods for accurate/efficient computation of large-scale flow with evolving phase boundaries

(involves inversion of a nonlinear elliptic problem with unknown phase boundaries)