Dynamics of inertial particles and dynamical systems (I)

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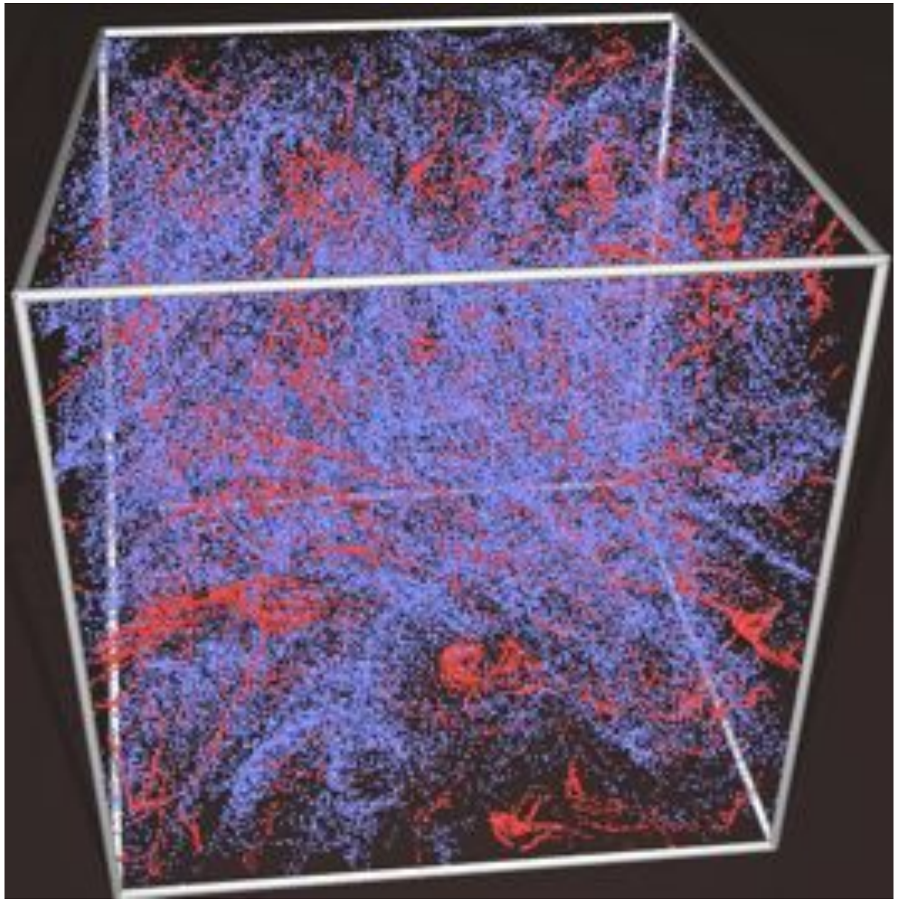


Goal

Understanding general properties of inertial particles advected by fluid flows from a dynamical systems point of view

Outline

Lecture 1:

- Model equations for inertial particles & introductory overview of dynamical systems ideas and tools

Lecture 2:

- Application of dynamical systems ideas and tools (lecture 1) to inertial particles for characterizing clustering

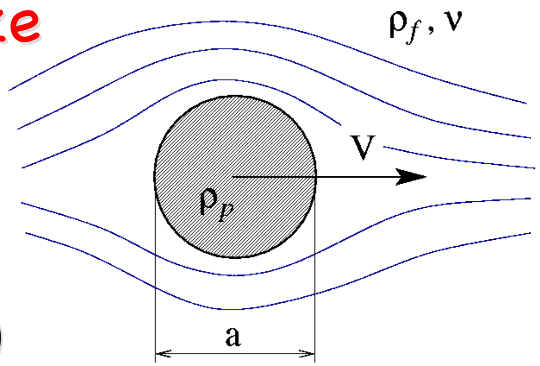
Two kinds of particles

Tracers= same as fluid elements

- same density of the fluid $\rho_p = \rho_f$
- point-like
- same velocity of the underlying fluid velocity

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{X}(t), t)$$

Inertial particles= mass impurities of finite size

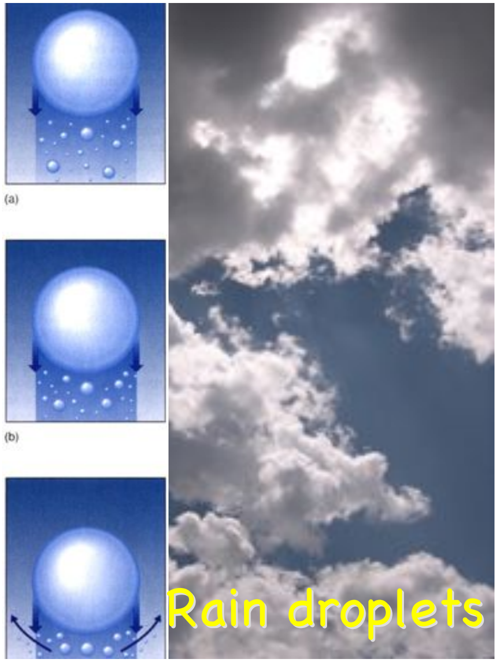


- density different from that of the fluid $\rho_p \neq \rho_f$
- finite size
- friction (Stokes) and other forces should be included
- shape may be important (we assume spherical shape)
- velocity mismatch with that of the fluid

Simplified dynamics under some assumptions

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{V}$$
$$\frac{d\boldsymbol{V}}{dt} = \boldsymbol{F}(\boldsymbol{V}, \boldsymbol{u}(\boldsymbol{X}(t), t), a, \nu, \ldots)$$

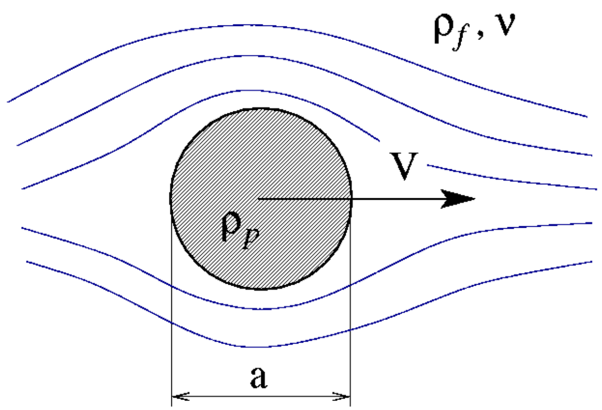
Relevance of inertial particles



Finite-size & mass impurities in fluid flows

…and Pyroclasts

Particle Dynamics



Single particle

Particle: rigid sphere, radius a, mass m_p; passive => no feedback on the fluid

Fluid around the particle: Stokes flow

$$\boxed{\frac{a(u-V)}{\nu} \ll 1 \qquad a \ll \eta}$$

$$m_p \frac{dV_i}{dt} = (m_p - m_f) g_i + m_f \left.\frac{Du_i}{Dt}\right|_{\boldsymbol{X}(t)}$$

bouyancy

$$-6\pi a \mu \left[V_i(t) - u_i(\boldsymbol{X}(t), t) - \frac{1}{6} a^2 \left.\nabla^2 u_i\right|_{\boldsymbol{X}(t)} \right]$$

Stokes drag Faxen correction

$$-\frac{m_f}{2} \frac{d}{dt} \left[V_i(t) - u_i(\boldsymbol{X}(t), t) - \frac{1}{10} a^2 \left.\nabla^2 u_i\right|_{\boldsymbol{X}(t)} \right]$$

Added mass

$$-6\pi a \mu \int_0^t ds \left(\frac{d/ds \left[V_i(s) - u_i(\boldsymbol{X}(s), s) - \frac{1}{6} a^2 \left.\nabla^2 u_i\right|_{\boldsymbol{X}(s)} \right]}{\sqrt{\pi \nu (t-s)}} \right)$$

Basset memory term

Maxey & Riley (1983)

Auton et al (1988)

Simplified dynamics

prescibed fluid velocity field
(e.g. from Navier Stokes or random)

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{V}$$

$$\frac{d\boldsymbol{V}}{dt} = \beta\frac{D\boldsymbol{u}(\boldsymbol{X},t)}{Dt} + \frac{\boldsymbol{u}(\boldsymbol{X},t)-\boldsymbol{V}}{\tau_p} + (1-\beta)\boldsymbol{g}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}$$

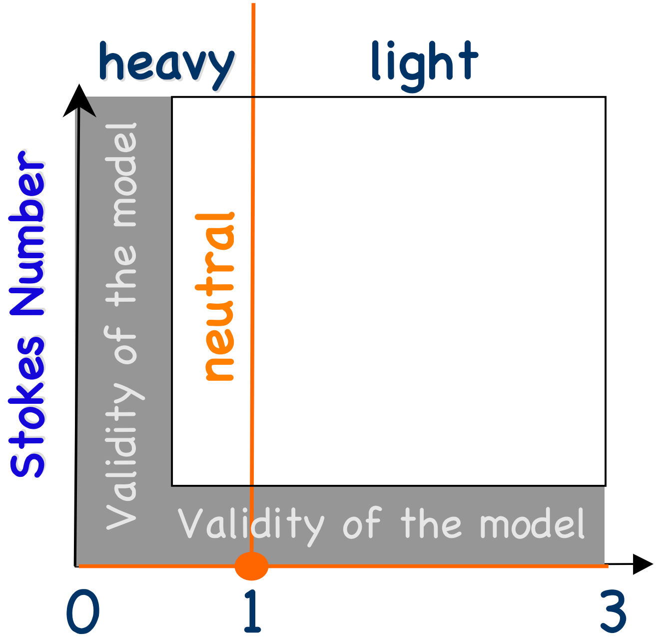
Stokes time $\tau_p = \dfrac{a^2}{3\nu\beta}$

Fastest fluid time scale

$$\tau_f = \tau_\eta = \frac{L}{U}Re^{-1/2}$$

$$\boxed{St = \frac{\tau_p}{\tau_f}}$$

two adimensional
control parameters St & β



Density contrast

$$\boxed{\beta = \frac{3\rho_f}{\rho_f + 2\rho_p}}$$

As a further simplification we will ignore gravity

Starting point of this lecture

Tracers

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(\boldsymbol{X}(t), t)$$

Inertial particles

$$\begin{aligned} \frac{d\boldsymbol{X}}{dt} &= \boldsymbol{V} \\ \frac{d\boldsymbol{V}}{dt} &= \beta \frac{D\boldsymbol{u}(\boldsymbol{X}, t)}{Dt} + \frac{\boldsymbol{u}(\boldsymbol{X}, t) - \boldsymbol{V}}{\tau_p} \end{aligned}$$

Let's forget that we are studying particles moving in a fluid! What do we know about a generic system of nonlinear ordinary differential equations?

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{g}(\boldsymbol{x})$$

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$$

$$\boldsymbol{g} = (g_1, g_2, \ldots, g_d)$$

Dynamical systems

- $$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1(t), x_2(t), \cdots, x_d(t)) \\ \vdots \\ \dfrac{\mathrm{d}x_d}{\mathrm{d}t} = f_d(x_1(t), x_2(t), \cdots, x_d(t)) \end{cases} \qquad \frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x})$$ Autonomous ODE

d+1

- $$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t)$$ non-autnonomous ODE $x_{d+1} = t$ e $f_{d+1} = 1$

- x(t+1)=f(x(t)) Maps (discrete time)

- $$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \nu \Delta \boldsymbol{v} + \boldsymbol{f}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$ PDEs d->∞

Examples of ODEs

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\sigma X + \sigma Y$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -XZ + rX - Y$$
$$\frac{\mathrm{d}Z}{\mathrm{d}t} = XY - bZ\,.$$

Lorenz model

d=3

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

From Mechanics
(Hamiltonian systems)

i=1,N => d=2N

with

$$x_i = q_i \quad x_{i+N} = p_i \qquad \boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})$$

$$f_i = \frac{\partial H}{\partial p_i} \quad f_{i+N} = -\frac{\partial H}{\partial q_i} \qquad \boldsymbol{f} = (\nabla_p H, -\nabla_q H)$$

$$\Rightarrow \quad \frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x})$$

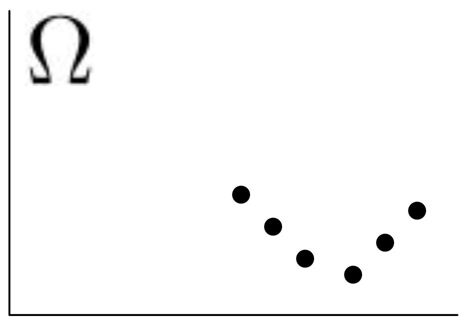
Some nomenclature

The space spanned by the system variables is called phase space

Exs: N particles $\Gamma \equiv \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N\}$ (2xd)xN dimensions

Lorenz model $\Omega \equiv \{X, Y, Z\}$ 3 dimensions

For tracers the phase space coincides with the real space
For inertial particles the phase space accounts for both particle's position and velocity



A point in the phase space identifies the system state

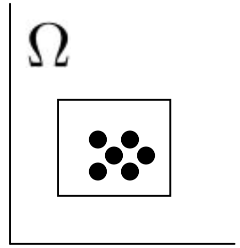
A trajectory is the time succession of points in the phase space

We can distinguish two type of dynamics in phase-space

Conservative & dissipative

Given a set of initial conditions distributed with a given density

$$\rho(x,0) \quad \text{with} \quad \int_\Omega d\boldsymbol{x}\,\rho(\boldsymbol{x},0)=1$$



Given $\dot{\boldsymbol{x}}=\boldsymbol{f}$ how does $\rho(\boldsymbol{x},t)$ evolve?

$$\partial_t\rho+\nabla\cdot(\boldsymbol{f}\rho)=\partial_t\rho+\boldsymbol{f}\cdot\nabla\rho+\rho\nabla\cdot\boldsymbol{f}=0$$

Continuity equation ensuring $\int_\Omega d\boldsymbol{x}\,\rho(\boldsymbol{x},t)=1$

Conservative dynamical systems (Liouville theorem)

$$\boxed{\nabla\cdot\boldsymbol{f}=0}$$

Density is conserved along the flow as in incompressible fluids ==>phase space volumes are conserved

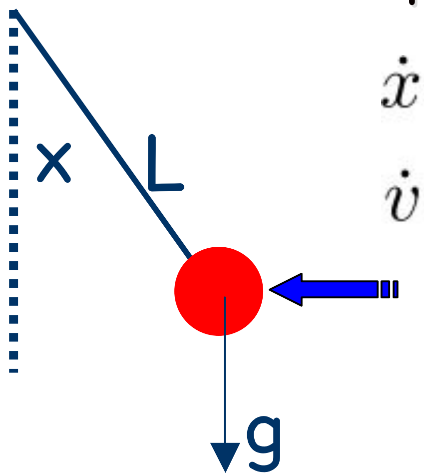
Dissipative dynamical systems

$$\boxed{\nabla\cdot\boldsymbol{f}<0}$$

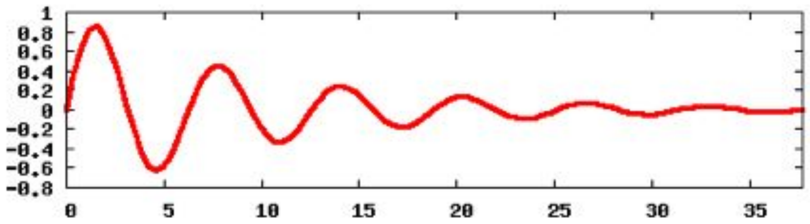
Volumes are exponentially contracted as the integral of the density is constant => density has to grow somewhere

Examples of dissipative systems

The harmonic pendulum with friction

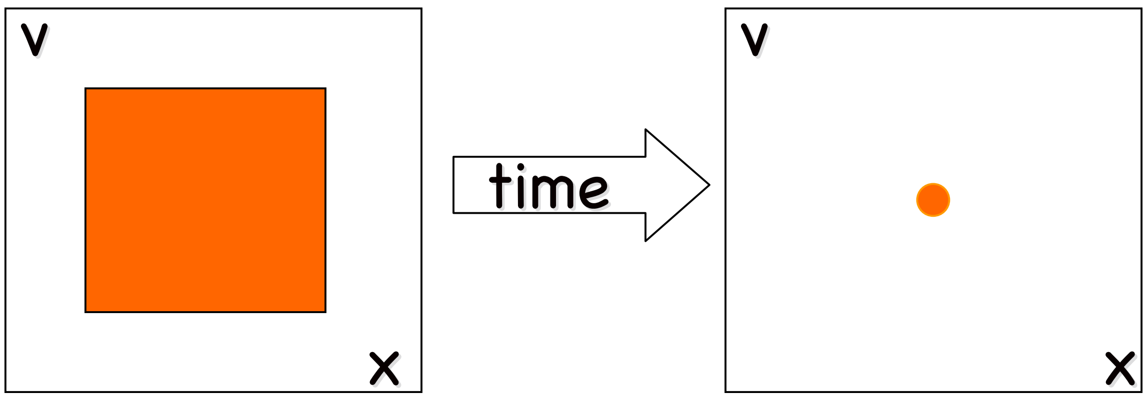


$$\dot{x} = v$$
$$\dot{v} = -\frac{g}{L}x - \gamma v$$



$$\boldsymbol{f} = (v, -gx/L - \gamma v) \quad \rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{f} = -\gamma < 0$$

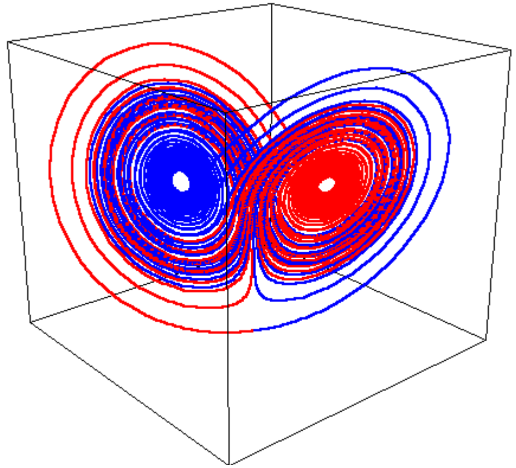
Phase-space volumes are exponentially contracted to the point (x,v)=(0,0) which is an attractor for the dynamics



The existence of an attractor (set of dimension smaller than that of the phase space where the motions take place) is a generic feature of dissipative dynamical systems

Lorenz model

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\sigma X + \sigma Y$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -XZ + rX - Y$$
$$\frac{\mathrm{d}Z}{\mathrm{d}t} = XY - bZ\,.$$



$$\frac{\partial f_i}{\partial x_j} = \mathbb{L}_{ij}(\boldsymbol{x})$$

Stability Matrix

$$\mathbf{L} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (r-Z) & -1 & -X \\ Y & X & -b \end{pmatrix}.$$

$$\boxed{\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\partial}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}t} + \frac{\partial}{\partial Z}\frac{\mathrm{d}Z}{\mathrm{d}t} = \mathrm{Tr}\,(\mathbb{L}) = -(\sigma + b + 1) < 0}$$

b, r, σ positive

attractors can be strange objects

Inertial particles have a dissipative dynamics

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{V}$$
$$\frac{d\boldsymbol{V}}{dt} = \beta\frac{D\boldsymbol{u}(\boldsymbol{X},t)}{Dt} + \frac{\boldsymbol{u}(\boldsymbol{X},t)-\boldsymbol{V}}{\tau_p}$$

$$\boldsymbol{f} = (\boldsymbol{V}, \beta D_t \boldsymbol{u}(\boldsymbol{x},t) + \tfrac{\boldsymbol{u}-\boldsymbol{V}}{St})$$

$$\frac{\partial f_i}{\partial x_j} = \mathbb{L}_{ij}(\boldsymbol{x})$$
$$\sigma_{ij} = \partial_j u_i$$

$$\mathbb{L} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \beta D_t \sigma + \frac{\sigma}{\tau_p} & -\frac{\mathbb{I}}{\tau_p} \end{pmatrix}$$

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{f} = Tr(\mathbb{L}) = -\frac{d}{\tau_p} < 0}$$

Uniform contraction in phase space
as in Lorenz model

Examples of conservative systems

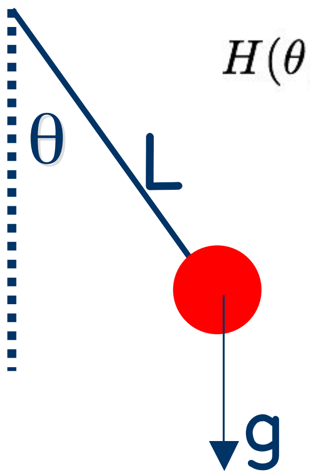
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Hamiltonian systems are conservative, but the reverse is not true

$$\boxed{\nabla \cdot \boldsymbol{f} = \sum_i \frac{\partial \dot{q}_i}{\partial p_i} + \frac{\partial \dot{p}_i}{\partial q_i} = \sum_i \frac{\partial^2 H}{\partial p_i \partial q_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0}$$

Nonlinear pendulum



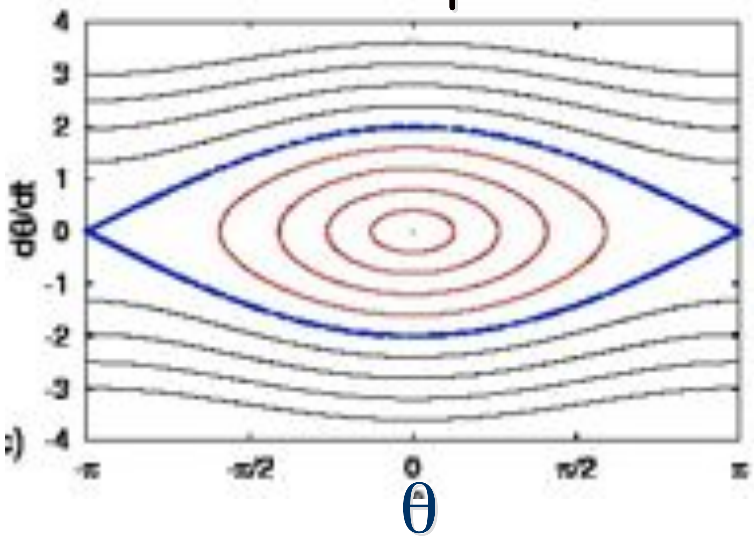
$$H(\theta, \dot{\theta}) = \frac{1}{2} m L^2 \dot{\theta}^2 + mgL(1 - \cos\theta)$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

$$q = \theta \qquad \dot{q} = p$$

$$p = \dot{\theta} \quad \Rightarrow \quad \dot{p} = -\frac{g}{L}\sin q$$

Phase space

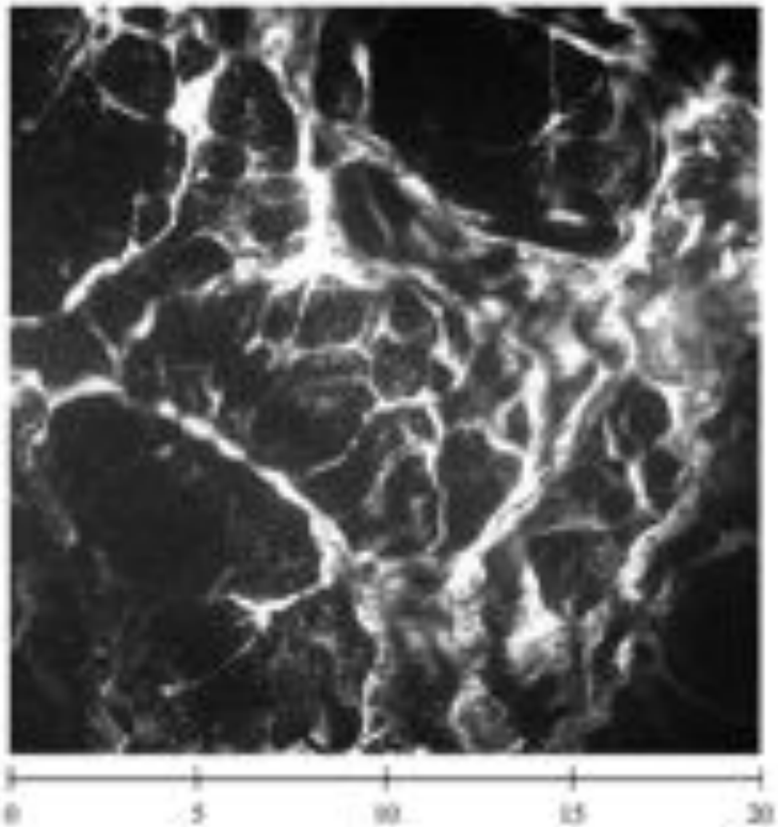


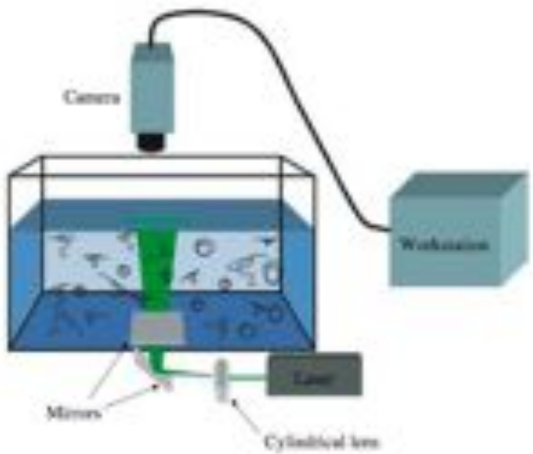
In conservative systems there are no attractors

Tracers

Incompressible flows: conservative $\dot{\boldsymbol{X}} = \boldsymbol{u}(\boldsymbol{X}, t) \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$

Compressible flows: dissipative $\dot{\boldsymbol{X}} = \boldsymbol{u}(\boldsymbol{X}, t) \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} < 0$





E.g. tracers on the surface of a 3d incompressible flows visualization of an attractor

John R Cressman[1], Jahanshah Davoudi[2], Walter I Goldburg[1] and Jörg Schumacher[2]

Basic questions

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x})$$

- Given the initial condition x(0), when does exists a solution? I.e. which properties f(x) must satisfy?

- When solutions exist, which type of solutions are possible and what are their properties?

Theorem of existence and uniqueness

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Re^d$$

with x(0) given

if f is continuous with the Lipschitz condition
(essentially if f is differentiable)

$$||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|| \leq K||\boldsymbol{x} - \boldsymbol{y}||$$

The solution exists and is unique

Counterexample

$$\frac{dx}{dt} = \frac{3}{2}x^{1/3}$$

Non-Lipschitz in x=0

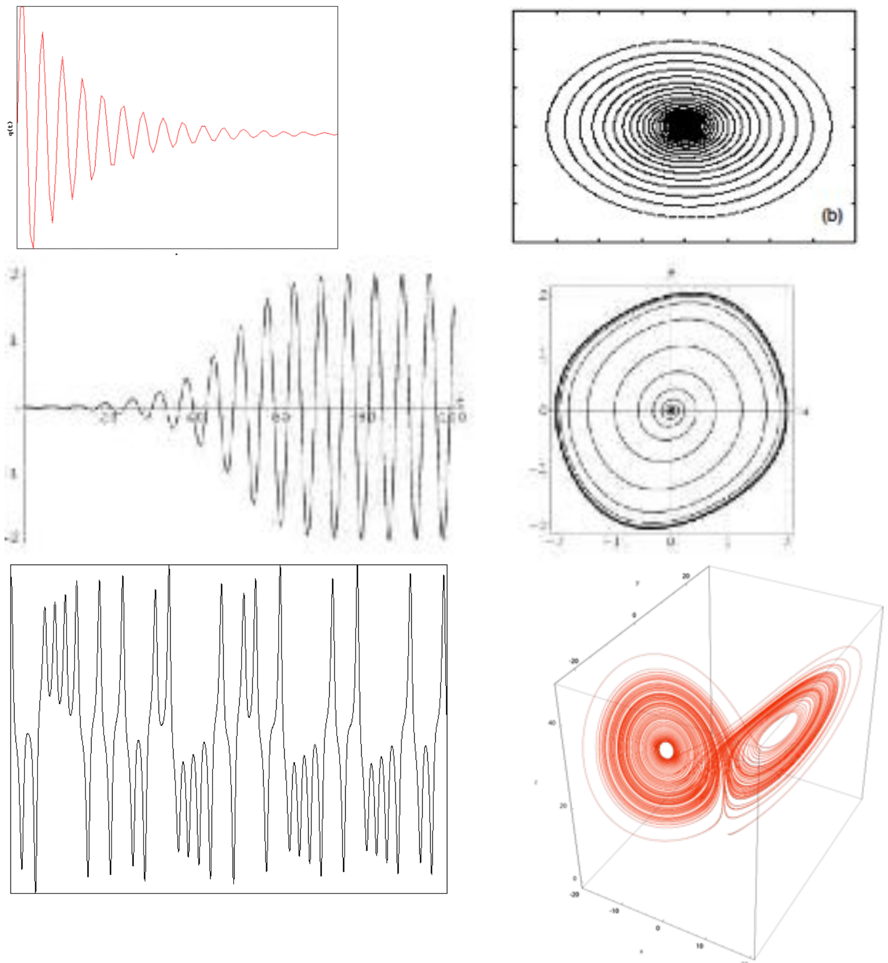
with x(0)=0 two solutions $x(t) = 0$ & $x(t) = t^{3/2}$

Which kind of solutions?

In dissipative systems motions converge onto an attractor and can be regular or irregular

Regular

Irregular



Attracting fixed point

(pendulum with friction)

Limit cycle

(asymptotically periodic)

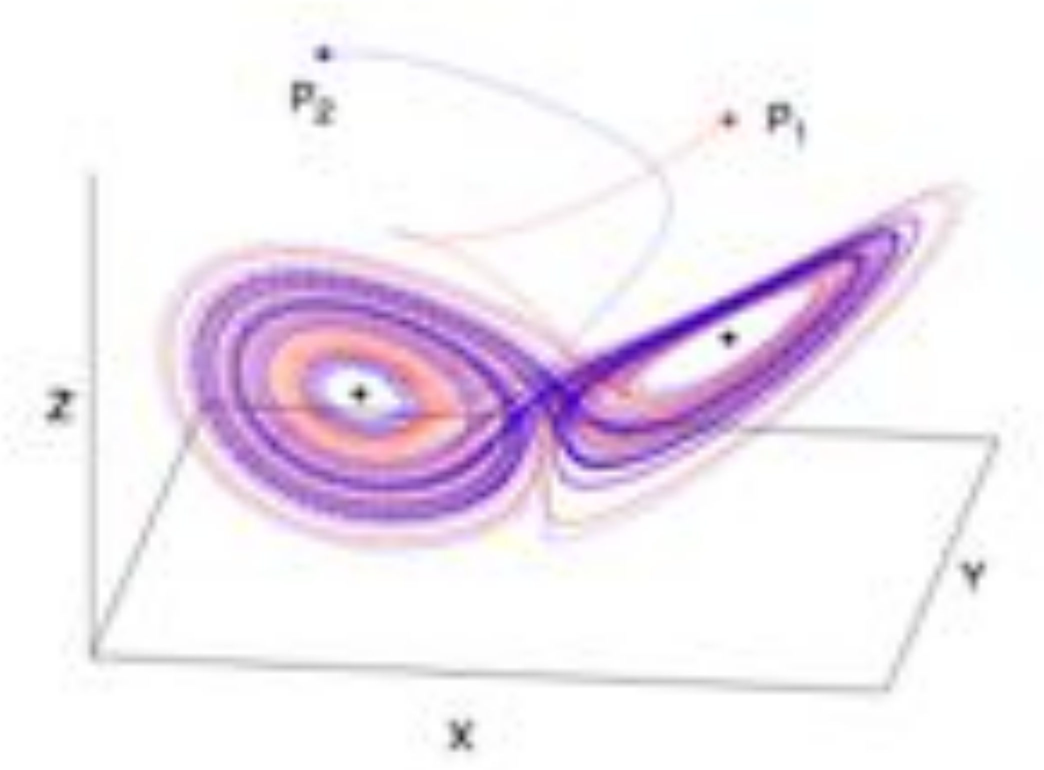
(Van der Pool oscillator)

Strange Attractors

(Lorenz model)

Different kind of motion can be present in the same system changing the parameters

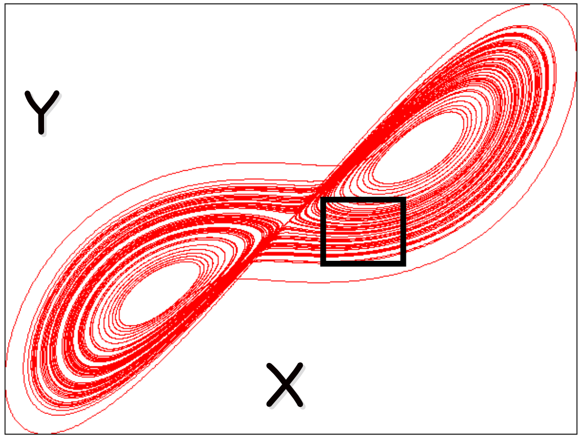
Strange attractors

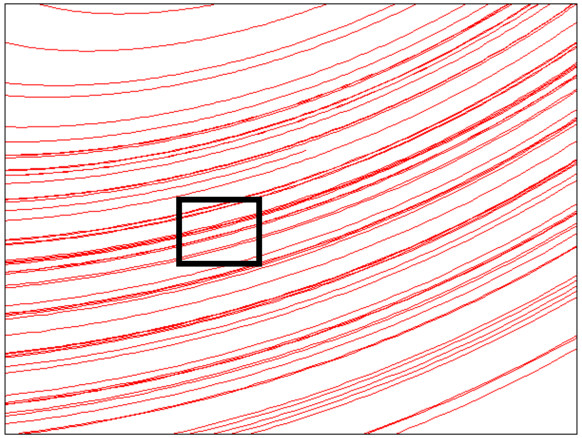


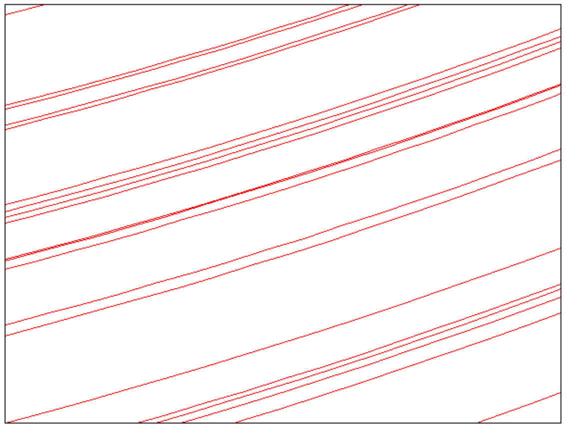
Typically, the dynamics on the strange attractor is ergodic
averages of observables do not depend on the initial conditions
(difficult to prove!)

Strange attractors

Have complex geometries





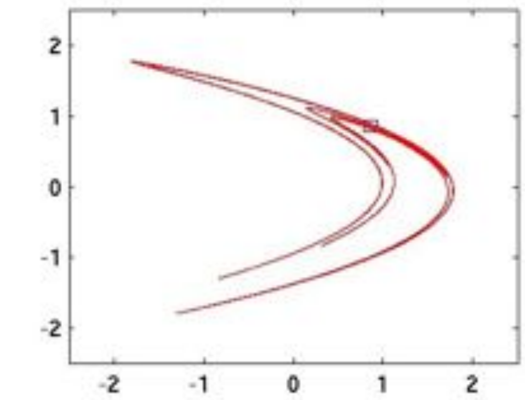


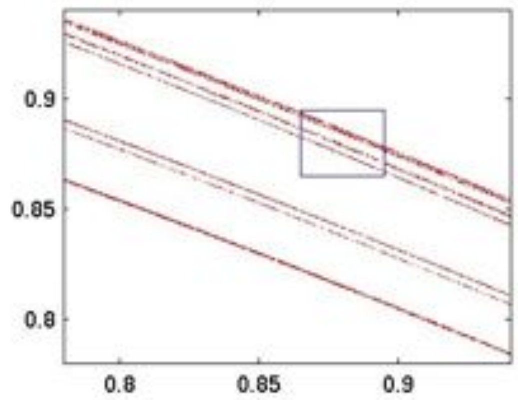
Non-Smooth geometries
Self-similarity
The points of the trajectory distribute in a very singular way

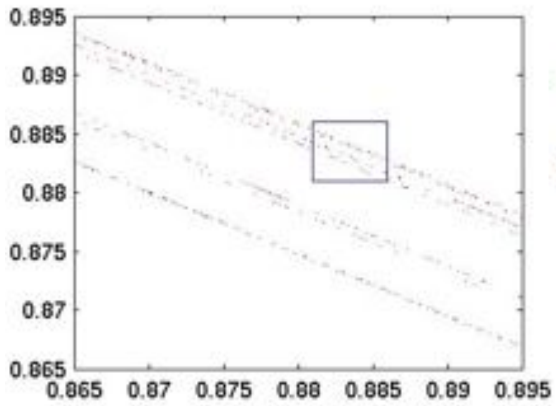
These geometries can be analyzed using tools and concepts from (multi-)fractal objects

Fractality is a generic feature

Of the strange attractors







Hénon map

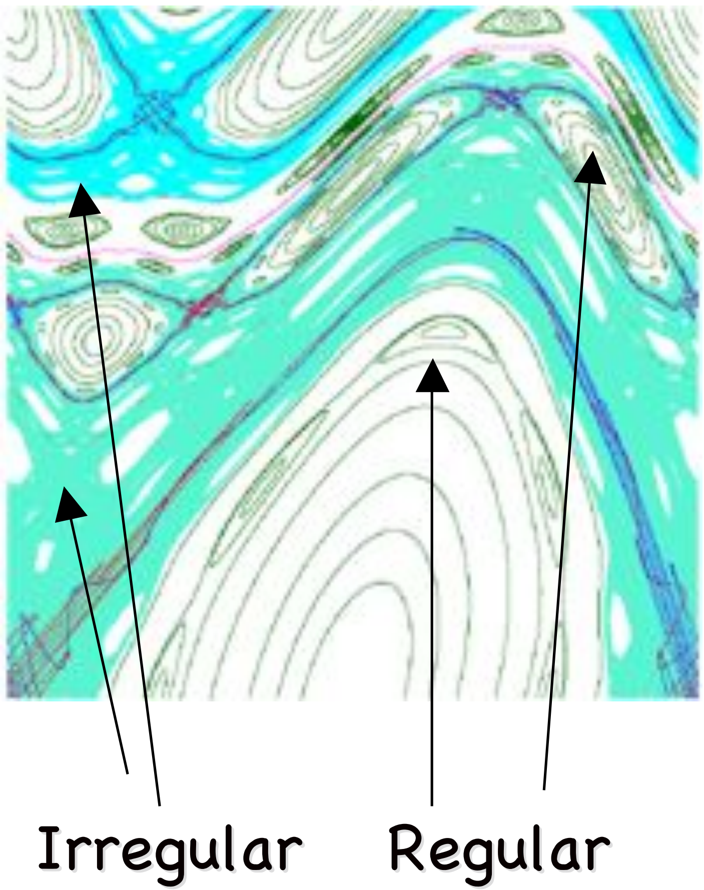
$$x_1(n+1) = x_2(n) + 1 - a x_1^2(n)$$

$$x_2(n+1) = b x_1(n)\,,$$

a=1.4 b=0.3

Which kind of solutions?

In conservative systems motions can take place in all the avalaible phase space and can be regular or irregular.
Often coexistence of regular and irregular motions in different regions depending on the initial condition (non-ergodic)

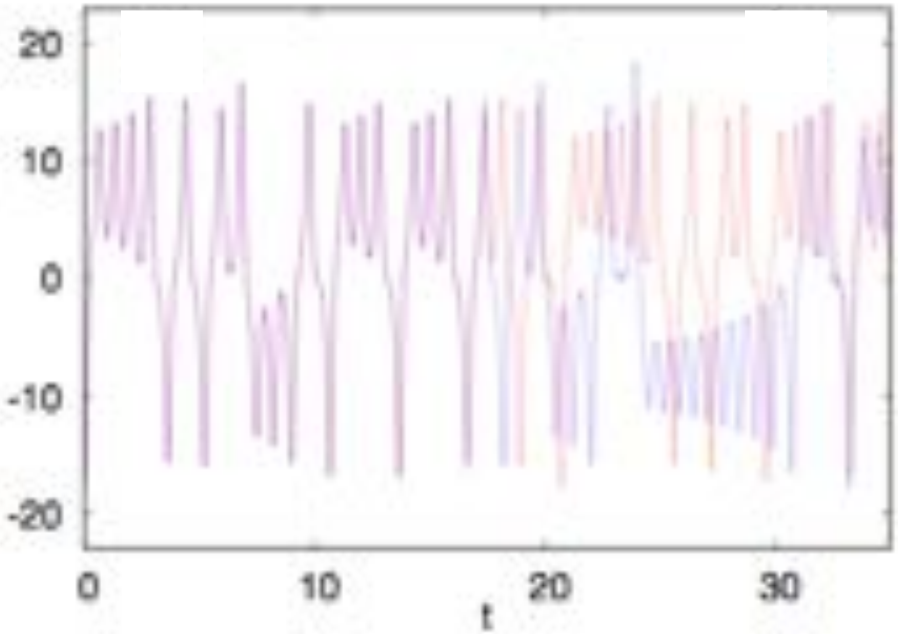


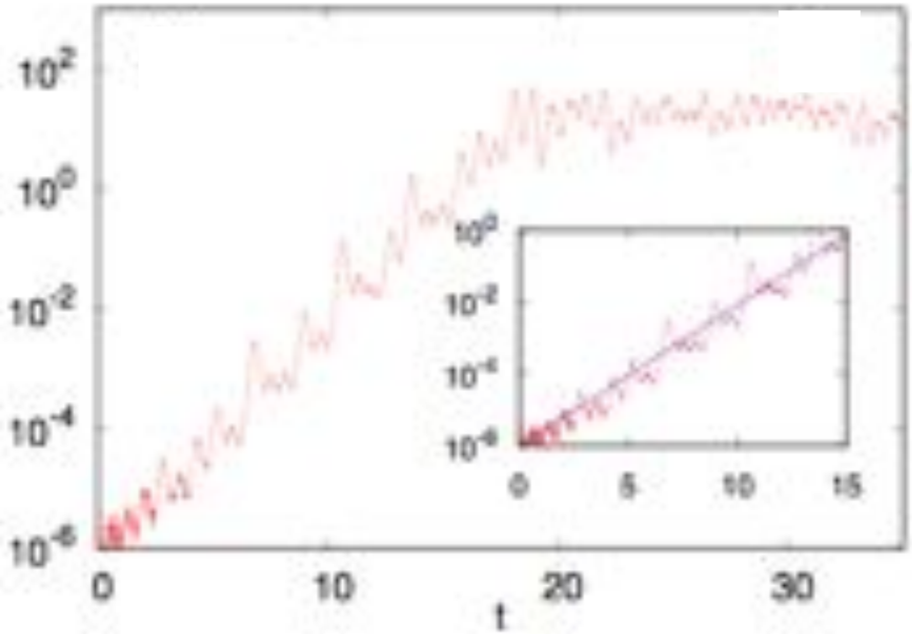
The onset of the mixed regime can be understood through KAM theorem

In turbulence, tracers, which are conservative, have irregular motions for essentially all initial conditions and they visit all the avalaible space filling it uniformly
(ergodicity & mixing hold)

Sensitive dependence on initial conditions

In both dissipative and conservative systems, irregular trajectories display sensitive dependence on initial conditions which is the most distinguishing feature of chaos





$\boldsymbol{R}(0) = (X(0), Y(0), Z(0))$
$\boldsymbol{R}'(0) = \boldsymbol{R}(0) + \boldsymbol{\Delta}(0)$

Exponential separation of generic infinitesimally close trajectories

$$|\boldsymbol{\Delta}(t)| = |\boldsymbol{R}(t) - \boldsymbol{R}'(t)| \approx |\boldsymbol{\Delta}(0)| \exp(\lambda t)$$

How to make these observations quantitative?

We focus on dissipative systems
which are relevant to inertial particles

We need:

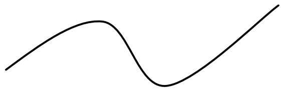
1 To characterize the geometry of strange attactors:
fractal and generalized dimensions

2 To characterize quantitatively the sensitive on initial conditions: Characteristic Lyapunov exponents

How to characterize fractals?

Simple objects can be characterized in terms of the topological dimension d_T

Point • $d_T=0$

Curve  $\Longleftrightarrow$ $\{x\}\subset \mathbb{R}^1$ $d_T=1$

Surface  $\Longleftrightarrow$ $\{x,y\}\subset \mathbb{R}^2$ $d_T=2$

But d_T seems to be unsatisfatory for more complex geometries

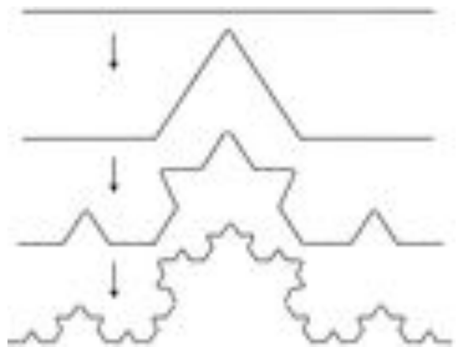
Cantor set

$d_T=0$

(disjoined points)

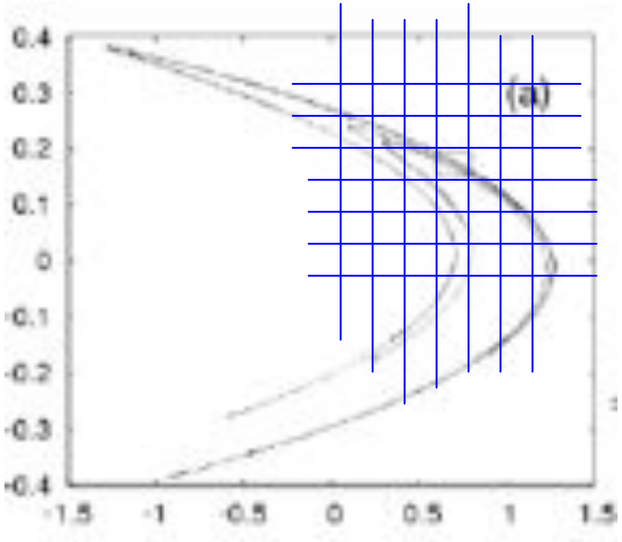
Koch curve

$d_T=1$



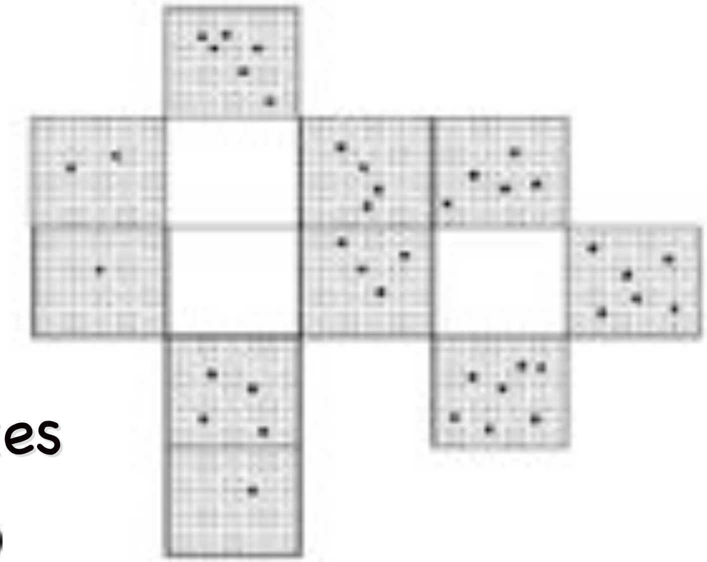
Box counting dimension

Another way to define the dimension of an object



Grey boxes

Contains at least 1 point



$N(\ell)$# grey boxes

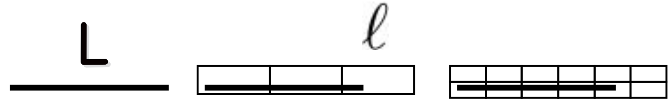
$$N(\ell) \propto \ell^{-D}$$

$$\boxed{D = -\lim_{\ell \to 0} \frac{\ln N(\ell)}{\ln \ell}}$$

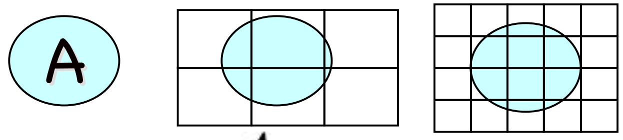
Mathematically more rigorous is to use the Hausdorff dimension equivalent to box counting in most cases.

Box counting dimension

For regular objects the box counting dimension coincides with the topological one

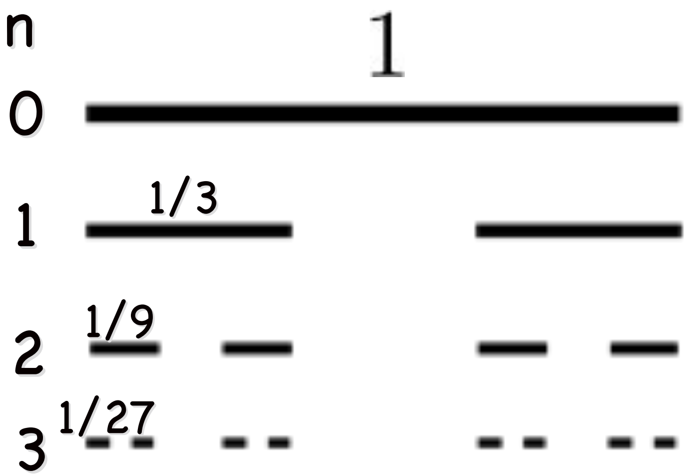


$$N(\ell) \approx \frac{L}{\ell} \qquad D = d_T = 1$$



$$N(\ell) \approx \frac{A}{\ell^2} \qquad D = d_T = 2$$

for more complex objects?



ℓ	$N(\ell)$
1	1
$1/3$	2^1
$1/3^2$	2^2
$1/3^3$	2^3

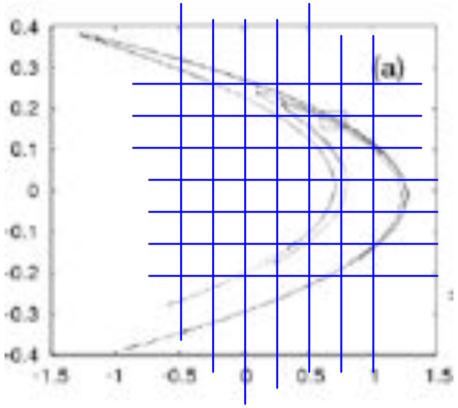
$$\ell = 3^{-n} \qquad N(n) = 2^n$$

$$N(\ell) = \ell^{-\ln(2)/\ln(3)}$$

$$\boxed{D_F = -\lim_{\ell \to 0} \frac{\ln N(\ell)}{\ln \ell} = \frac{\ln 2}{\ln 3} > 0 = d_T}$$

For fractal object the box counting dimension is larger than the topological one and is typically a non-integer number

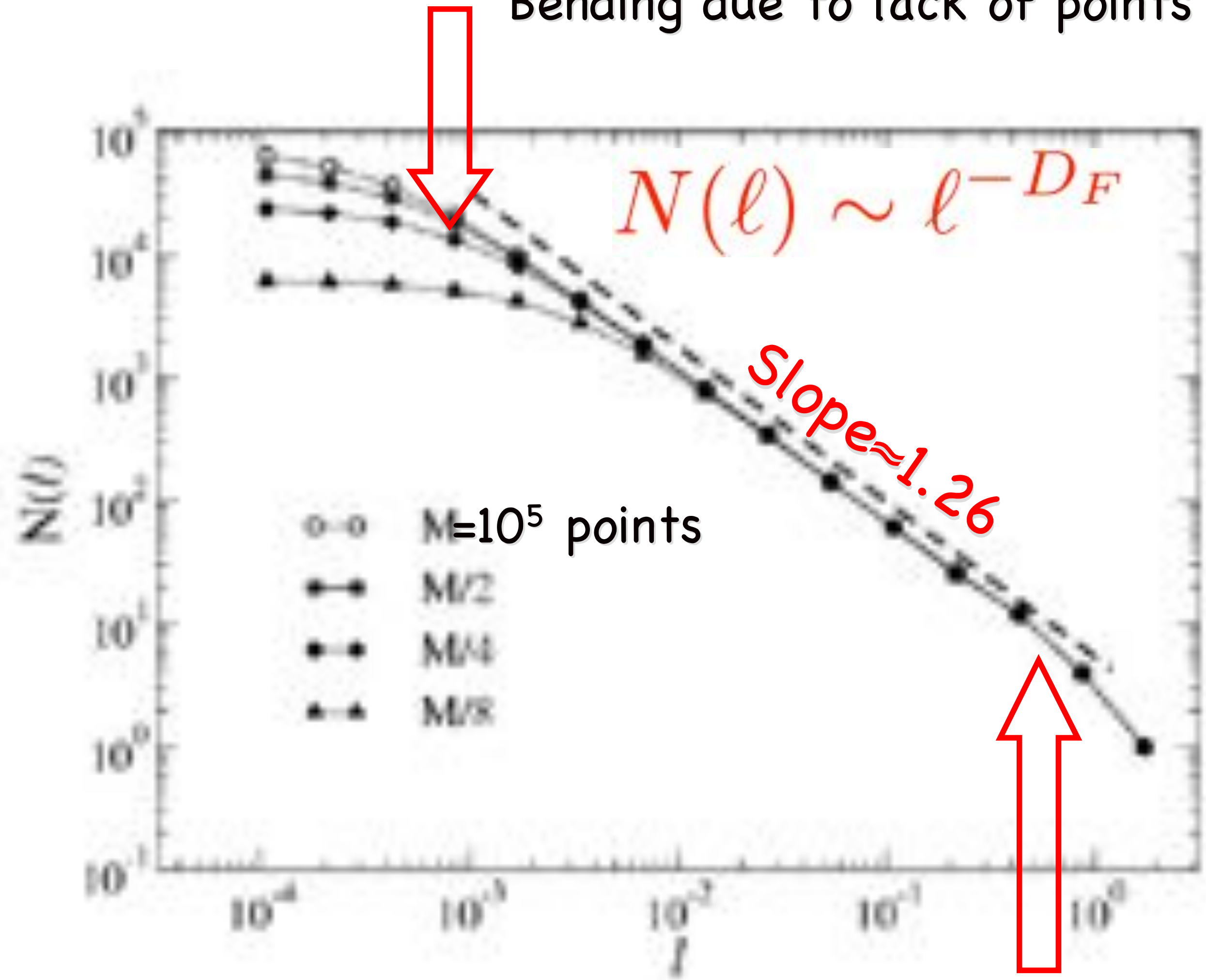
Hénon attractor



D_F=1.26

For the Lorenz attractor D_F=2.05

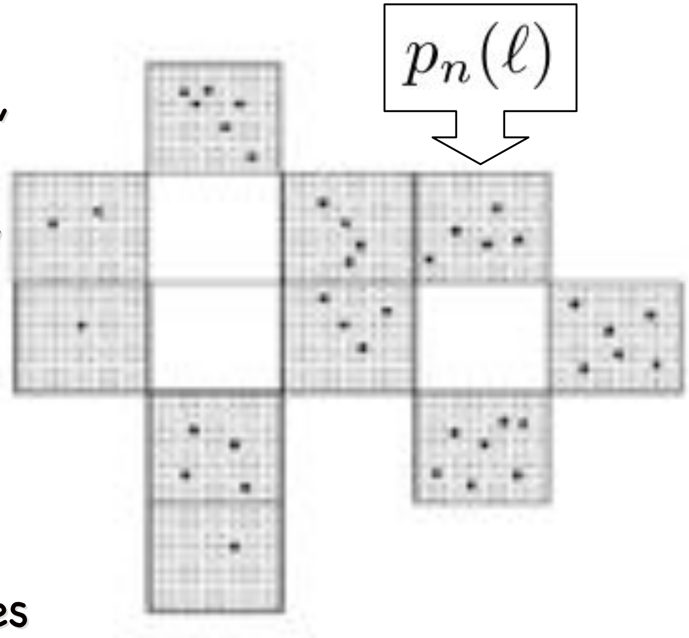
Bending due to lack of points



Effect of finite extension

Multifractals: Generalized dimensions

The fractal dimension does not account for fluctuations, characterizes the support of the object but does not give information on the measure properties i.e. the way points distribute on it.



Local fractal dimension

$$p_n(\ell) \sim \ell^{\alpha_n}$$

Sum over all occupied boxes

$$\mathcal{M}_\ell(q) = \sum_{k=1}^{N(\ell)} [p_k(\ell)]^q = \sum_{k=1}^{N(\ell)} [p_k(\ell)]^{q-1} p_k(\ell) = \langle [p_k(\ell)]^{q-1} \rangle$$

$$\mathcal{M}_\ell(0) = N(\ell) = \ell^{-D_F}$$

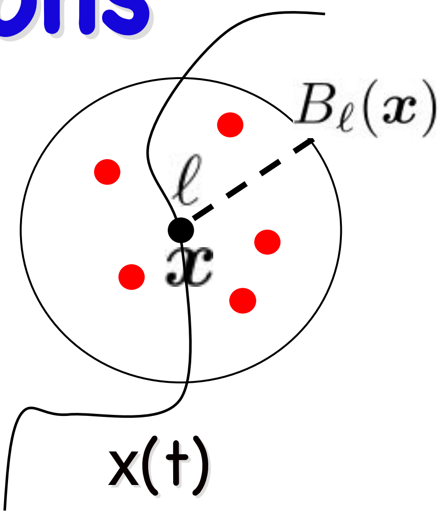
$$\mathcal{M}_\ell(q) \sim \ell^{(q-1)D(q)}$$

$$\boxed{D(q) = \frac{1}{q-1} \lim_{\ell \to 0} \frac{\ln \mathcal{M}_\ell(q)}{\ln \ell}}$$

D(q) characterize the fluctuations of the measure on the attractor

Generalized dimensions

$$\boxed{\langle [p_{B_\ell(\boldsymbol{x})}]^q \rangle \sim \ell^{qD(q+1)}}$$



$D(0) = D_F$ Fractal dimension

$$D(1) = \lim_{\ell \to 0} \frac{\sum_{n=0}^{N(\ell)} p_n(\ell) \ln p_n(\ell)}{\ln \ell}$$ Information dimension

$D(2) = D_{corr}$ Correlation dimension $P_2(||\boldsymbol{x}_1 - \boldsymbol{x}_2|| < r) \sim r^{D(2)}$

the smaller D(2) the larger the probability

$D(n)$ n integer: controls the probability to find n particles in a ball of size r

$$D(q) \le D(p) \quad for \quad q > p$$

In the absence of fluctuations (pure fractals) $D(q)=D(0)=D_F$

Characteristic Lyapunov exponents

Infinitesimally close trajectories separate exponentially

Linearized dynamics $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t)) \Longrightarrow \dot{\delta x_i} = \sum_{j=1}^{d} \partial_j f_i(\boldsymbol{x}(t))\delta x_j$

d=1

$$\delta x(t) = \delta x(0) e^{\int_0^t df(x(s))\,ds} = W(0,t)\delta x(0)$$

Law large numbers — ergodicity

$$\gamma(x_0,t) = \frac{1}{t}\ln\frac{\delta x(t)}{\delta x(0)} = \frac{1}{t}\int_0^t d_x f(s)\,ds \xrightarrow[t\to\infty]{} \langle d_x f\rangle = \lambda(x_0) = \lambda$$

Finite time Lyapunov exponent

Lyapunov exponent

$$|\delta x(t)| \sim |\delta x(0)| e^{\lambda t}$$

d>1

$$\delta\boldsymbol{x}(t) = \mathbb{W}(0,t)\delta\boldsymbol{x}(0)$$

$\Uparrow$

Evolution matrix (time ordered exponential)

We need to generalize the d=1 treatment to matrices (Oseledec theorem (1968))

Characteristic Lyapunov exponents

$$\delta\boldsymbol{x}(t) = \mathbb{W}(0,t)\delta\boldsymbol{x}(0) \qquad \Big[\underbrace{\mathbb{W}^{\dagger}(0,t)\mathbb{W}(0,t)}\Big]^{1/2} = \mathbb{V}(\boldsymbol{x}_0,t)$$

$$\mathbb{V}(\boldsymbol{x}_0,t) = \mathbb{Q}(\boldsymbol{x}_0,t)\mathbb{D}(\boldsymbol{x}_0,t)\mathbb{Q}^{\dagger}(\boldsymbol{x}_0,t) \Longleftarrow$$ Positive & symmetric

$$\mathbb{D}(\boldsymbol{x}_0,t) = \mathrm{diag}\{e^{t\gamma_1(\boldsymbol{x}_0,t)},\ldots,e^{t\gamma_d(\boldsymbol{x}_0,t)}\}$$

Finite time Lyapunov exponents

Oseledec--> $\gamma_i(\boldsymbol{x}_0,t) \xrightarrow[t\to\infty]{} \lambda_i(\boldsymbol{x}_0) = \lambda_i$ if ergodic

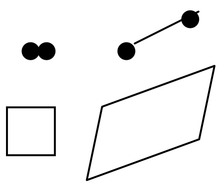
$$\boxed{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \quad \text{Lyapunov exponents}}$$

What is their physical meaning?

Characteristic Lyapunov exponents

λ_1 => growth rate of infinitesimal segments

$\lambda_1+\lambda_2$ => growth rate of infinitesimal surfaces



$L(t) = L(0)e^{\lambda_1 t}$

$A(t) = A(0)e^{(\lambda_1+\lambda_2)t}$

$\lambda_1+\lambda_2+\lambda_3$ => growth rate of infinitesimal volumes

: : :

$\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_d$ => growth rate of infinitesimal phase-space volumes

Chaotic systems have at least $\lambda_1>0$

Conservative systems $\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_d=0$

Dissipative systems $\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_d<0$

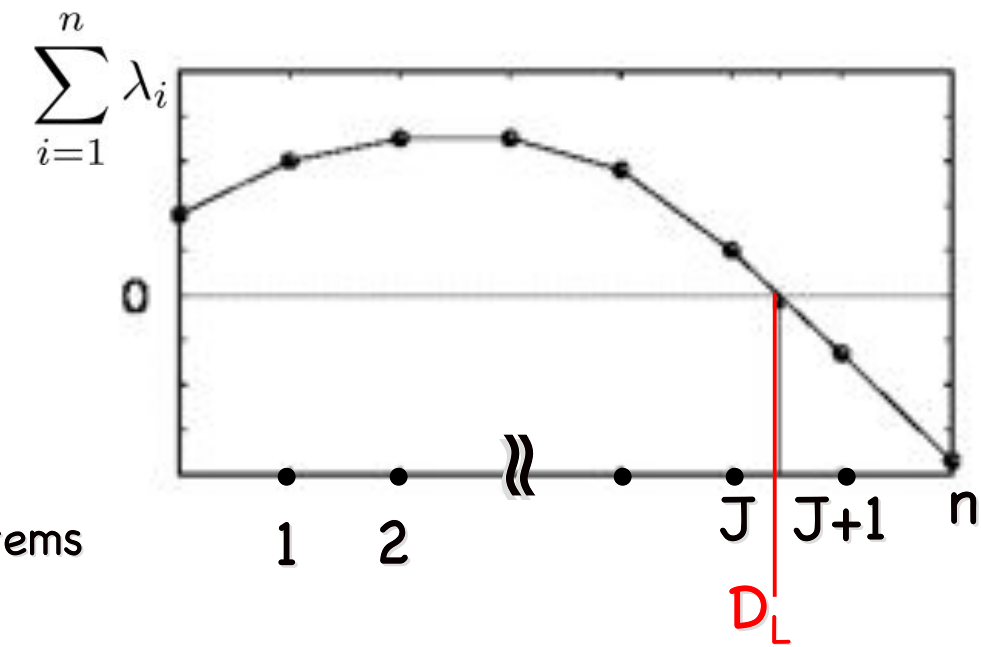
Lyapunov dimension

(Kaplan & Yorke 1979)

$$\boxed{D_L = J + \frac{\sum_{i=1}^{J} \lambda_i}{|\lambda_{J+1}|}}$$

One typically has $D(1) \le D_L$

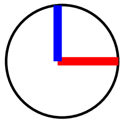
The equality holding for specific systems



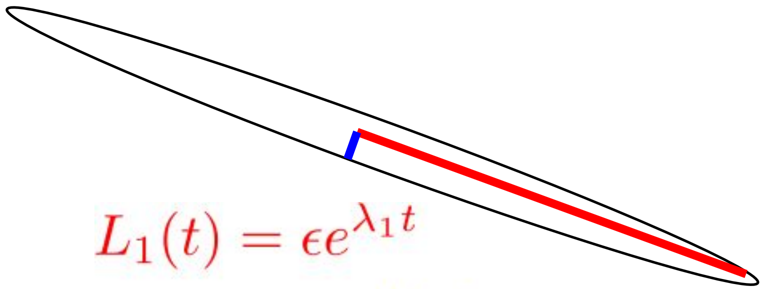
Lyapunov dimension

$$D_L = J + \frac{\sum_{i=1}^{J} \lambda_i}{|\lambda_{J+1}|}$$

Example
$\lambda_1>0$ $\lambda_2<0$



$$L_1(0) = L_2(0) = \epsilon$$



$$L_1(t) = \epsilon e^{\lambda_1 t}$$

$$L_2(t) = \epsilon e^{-|\lambda_2| t}$$

If we want to cover the ellipse with boxes of size $\ell = L_2$

Number of boxes

$$\ell^{-D_F} \approx N(\ell) \approx \frac{L_1}{L_2} \approx \ell^{-1-\lambda_1/|\lambda_2|}$$

$$\boxed{D_F = 1 + \frac{\lambda_1}{|\lambda_2|}}$$

Finite time Fluctuations of LE

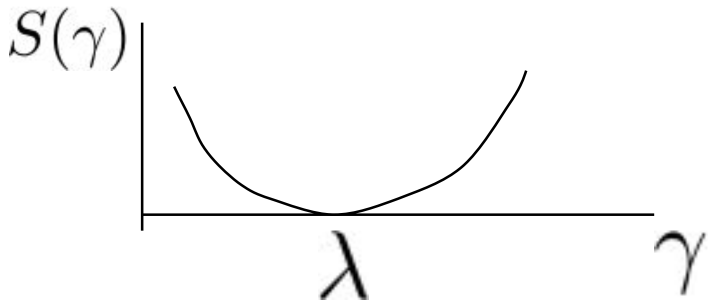
$$\delta\boldsymbol{x}(t) = \mathbb{W}(0,t)\delta\boldsymbol{x}(0) \qquad \left[\underbrace{\mathbb{W}^\dagger(0,t)\mathbb{W}(0,t)}\right]^{1/2} = \mathbb{V}(\boldsymbol{x}_0,t)$$

$$\mathbb{V}(\boldsymbol{x}_0,t) = \mathbb{Q}(\boldsymbol{x}_0,t)\mathbb{D}(\boldsymbol{x}_0,t)\mathbb{Q}^\dagger(\boldsymbol{x}_0,t) \quad \mathbb{D}(\boldsymbol{x}_0,t) = \mathrm{diag}\{e^{t\gamma_1(\boldsymbol{x}_0,t)},\ldots,e^{t\gamma_d(\boldsymbol{x}_0,t)}\}$$

$$\gamma_i(\boldsymbol{x}_0,t) \xrightarrow[t\to\infty]{} \lambda_i(\boldsymbol{x}_0)$$

For finite t γ's are fluctuating quantities, which can be characterized in terms of Large Deviation Theory

$$P(\gamma(t)=\gamma) \overset{t\to\infty}{\sim} e^{-tS(\gamma)}$$



In general

$$P_t(\gamma_1,\gamma_2,\ldots,\gamma_d) \overset{t\to\infty}{\sim} e^{-tS(\gamma_1,\gamma_2,\ldots,\gamma_d)}$$

The rate function S can be linked to the generalized dimensions (see e.g. Bec, Horvai, Gawedzki PRL 2004)

Summary

- Inertial particles & tracers in incompressible flows are examples of dissipative & conservative nonlinear dynamical systems

- Nonlinear dynamical systems are typically chaotic (at least one positive Lyapunov exponent)

- While chaotic and mixing conservative systems spread their trajectories uniformly distributing in phase space, dissipative systems evolve onto an attractor (set of zero volume in phase space) developing singular measures characterized by multifractal properties

Next lecture we focus on inertial particles their dynamics in phase space & clustering in position space

Reading list

Dynamical systems:

- J.P. Eckmann & D. Ruelle "Ergodic theory of chaos and strange attractors"
  RMP 57, 617 (1985)   [Very good review on dynamical systems]

Books (many introductory books e.g.):

- M. Cencini, F. Cecconi and A. Vulpiani
Chaos: from simple models to complex systems
World Scientific, Singapore, 2009
ISBN 978-981-4277-65-5

- E. Ott
  Chaos in dynamical systems
  Cambridge Universtity Press, II edition, 2002

Dynamics of inertial particles and dynamical systems (II)

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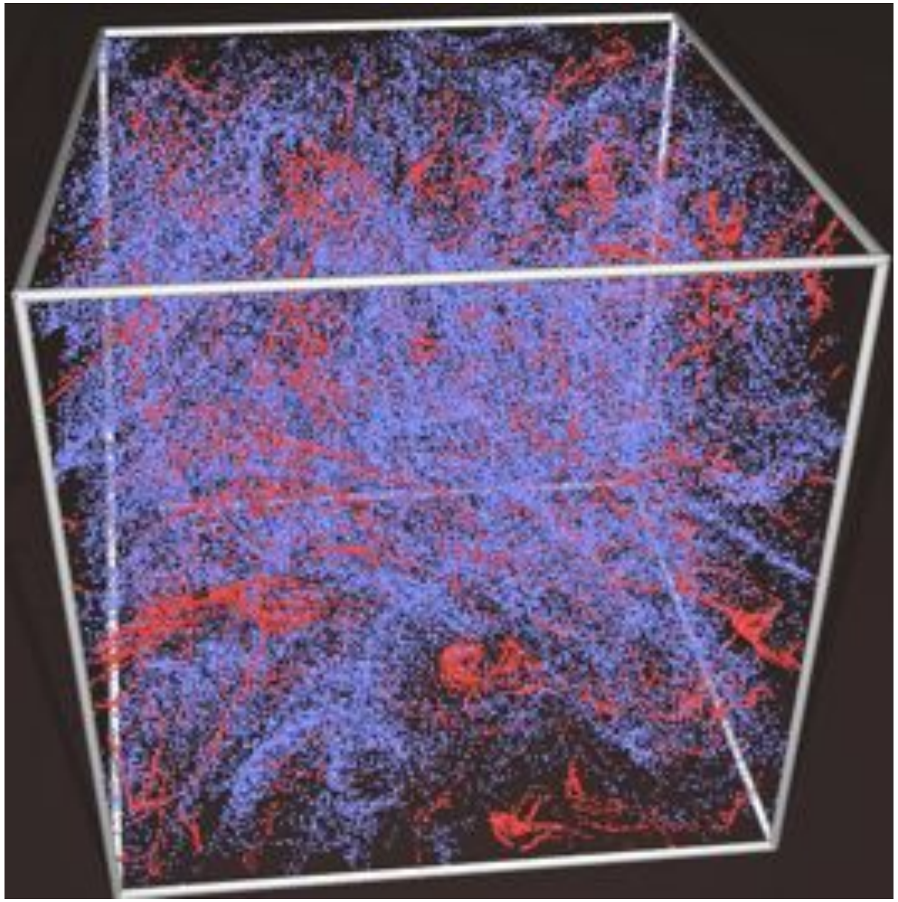
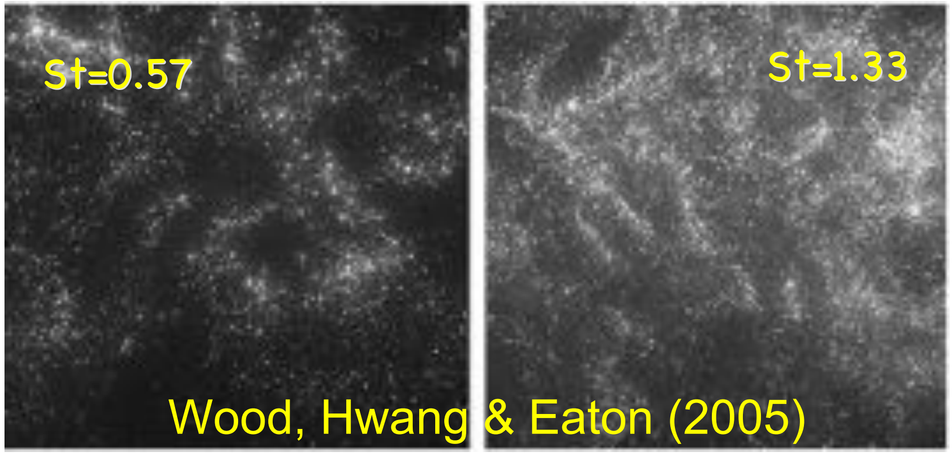


Goal

Dynamical and statistical properties of particles evolving in turbulence
focus on clustering observed in experiments



Clustering important for

- particle interaction rates by enhancing contact probability (collisions, chemical reactions, etc...)
- the fluctuations in the concentration of a pollutant
- the possible feedback of the particles on the fluid

We consider both turbulent & stochastic flows

Main interest dissipative range (very small scales)

Turbulent flows

In most natural and engeenering settings one is interested in particles evolving in turbulent flows i.e. solutions of the Navier-Stokes equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \Delta \boldsymbol{u} - \tfrac{1}{\rho_f} \nabla p + \boldsymbol{f} \qquad \nabla \cdot \boldsymbol{u} = 0$$

With large Reynolds number $Re = \frac{LU}{\nu} = \frac{\text{inertial t.}}{\text{dissipative t.}} \gg 1$

Basic properties

- K41 energy cascade with constant flux ε from large (~L) scale to the small dissipative scales (~η = Kolmogorov length scale)

- inertial range $\eta \ll r \ll L$ "almost" self-similar (rough) velocity field

$$\delta_r u = |u(x+r) - u(x)| \sim (\epsilon r)^{1/3}$$

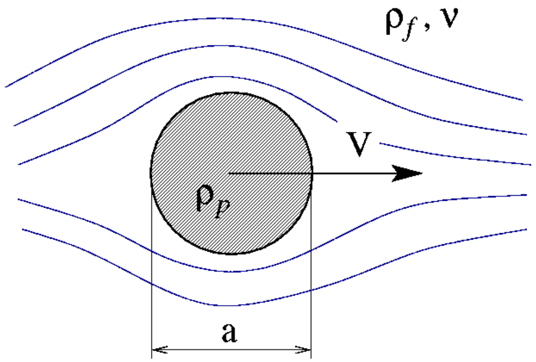
- dissipative range $r < \eta$ smooth (differentiable) velocity field

$$\delta_r u = |u(x+r) - u(x)| \propto r$$

Fast evolving scale: characteristic time ---> $\tau_f = \tau_\eta = \frac{L}{U} Re^{-1/2}$

(see Biferale lectures)

Simplified particle dynamics



Assumptions:

Small particles $a \ll \eta$
Small local Re $a|u-V|/\nu \ll 1$
No feedback on the fluid (passive particles)
No collisions (dilute suspensions)

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{V}$$
$$\frac{d\boldsymbol{V}}{dt} = \beta \frac{D\boldsymbol{u}(\boldsymbol{X})}{Dt} + \frac{\boldsymbol{u}(\boldsymbol{X},t) - \boldsymbol{V}}{St}$$

Stokes number

$$\boxed{St = \frac{\tau_p}{\tau_f}}$$

$\tau_p = \dfrac{a^2}{3\nu\beta}$ Stokes time

τ_f Fast fluid time scale

Density contrast

$$\boxed{\beta = \frac{3\rho_f}{\rho_f + 2\rho_p}}$$

$0 \le \beta < 1$ heavy
$\beta = 1$ neutral
$1 < \beta \le 3$ light

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{V}$$
$$\frac{d\boldsymbol{V}}{dt} = \frac{\boldsymbol{u}(\boldsymbol{X}(t),t) - \boldsymbol{V}}{St}$$

Minimal interesting model

Very heavy particle $\beta=0$
(e.g. water droplets in air $\beta=10^{-3}$)

Inertial Particles as dynamical systems

Particle in d-dimensional space

$$\begin{aligned}\dot{\boldsymbol{X}} &= \boldsymbol{V} \\ \dot{\boldsymbol{V}} &= \beta D_t \boldsymbol{u}(\boldsymbol{X}) + \frac{\boldsymbol{u}(\boldsymbol{X},t)-\boldsymbol{V}}{St}\end{aligned} \qquad \boldsymbol{X},\boldsymbol{V} \in \mathrm{R}^d$$

$\boldsymbol{u}(\boldsymbol{x},t)$

Differentiable at small scales (r<η)

Well defined dissipative dynamical system in 2d-dimensional phase-space

$$\dot{\boldsymbol{Z}} = \boldsymbol{F}(\boldsymbol{Z},t) \qquad \boldsymbol{F} = (\boldsymbol{V}, \beta D_t \boldsymbol{u}(\boldsymbol{x},t) + \tfrac{\boldsymbol{u}-\boldsymbol{V}}{St}) \qquad \boldsymbol{Z} = (\boldsymbol{X},\boldsymbol{V}) \in \mathrm{R}^{2d}$$

$\mathbb{L}_{ij} = \partial_j F_i$ Jacobian (stability matrix)

$\sigma_{ij} = \partial_j u_i$ Strain matrix

$$\mathbb{L} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \beta D_t \sigma + \frac{\sigma}{St} & -\frac{\mathbb{I}}{St} \end{pmatrix}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = Tr(\mathbb{L}) = -\frac{d}{St} < 0$$

constant phase-space contraction rate, i.e. phase-space Volumes contract exponentially with rate -d/St (similarly to Lorenz model)

Consequences of dissipative dynamics

- Motion must be studied in 2d-dimensional phase space
  (kinetic theory vs hydrodynamics)

- At large times particle trajectories will evolve onto an attractor (now dynamically evolving as $F(Z,t)$ depends on time)

- On the attractor particles distribute according to a singular (statistically stationary) density $\rho(X,V,t)$ whose properties are determined by the velocity field and parametrically depends on St & β

- Such singular density is expected to display multifractal properties; in particular, the fractal dimension of the attractor is expected to be smaller than the phase-space dimension $D_F<2d$

- The motion will be chaotic, i.e. at least one positive Lyapunov exponent

Two asymptotics

$$\begin{aligned}\dot{\boldsymbol{X}} &= \boldsymbol{V}\\ \dot{\boldsymbol{V}} &= \beta D_t \boldsymbol{u}(\boldsymbol{X}) + \frac{\boldsymbol{u}(\boldsymbol{X},t)-\boldsymbol{V}}{St}\end{aligned} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{F} = Tr(\mathbb{L}) = -\frac{d}{St}$$

$$\boxed{St = 0} \Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{F} = -\infty$$

Particle velocity relax to fluid one

$\dot{\boldsymbol{X}} = \boldsymbol{V} = \boldsymbol{u}(\boldsymbol{X},t)$ Becomes a tracer

Phase-space collapse to real space

where particles distribute uniformly

$D_F=d$

$$\boxed{St = \infty} \Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{F} = 0$$

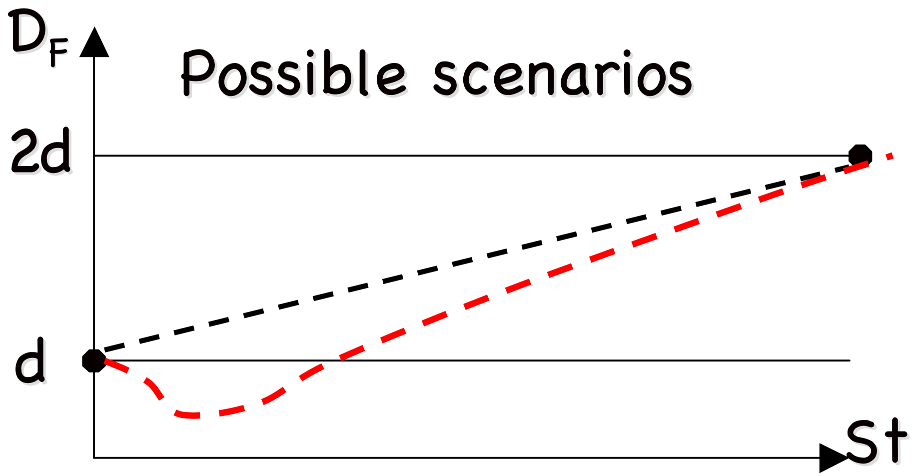
Particle velocity never relaxes

Ballistic limit, conservative dynamics

In 2d-dimensional phase space

Uniformly distributed in phase space

$D_F=2d$



Which scenario for D_F? (St<<1 limit)

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{u} = -\frac{1}{\rho_f}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f} \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

St<<1

$$D_t \boldsymbol{u}(\boldsymbol{X},t) \approx \dot{V} = \beta D_t \boldsymbol{u}(\boldsymbol{X},t) + \frac{\boldsymbol{u}(\boldsymbol{X},t) - \boldsymbol{V}}{St} \longrightarrow \boldsymbol{V} = \boldsymbol{u} + St(\beta - 1) D_t \boldsymbol{u}$$

(Maxey 1987; Balkovsky, Falkovich, Fouxon 2001)

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = St(\beta - 1)\boldsymbol{\nabla} \cdot (\boldsymbol{u}\boldsymbol{\nabla} \cdot \boldsymbol{u}) = St(\beta - 1)(S^2 - \Omega^2)$$

$$\boxed{\beta < 1 \quad S^2 > \Omega^2 \Longrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{V} < 0}$$

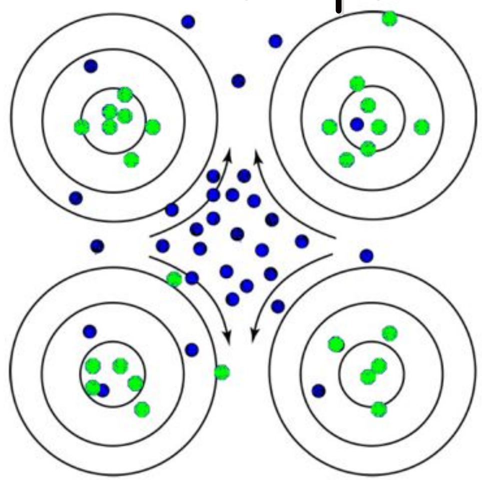
$$\boxed{\beta > 1 \quad \Omega^2 > S^2 \Longrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{V} < 0}$$

$$\sigma_{ij} = \frac{\partial u_i}{\partial x_j}$$

strain $\quad S_{ij} = \dfrac{\sigma_{ij} + \sigma_{ji}}{2}$

vorticity $\quad \Omega_{ij} = \dfrac{\sigma_{ij} - \sigma_{ji}}{2}$

d=2 example



• β<1 heavy
• β>1 light

Preferential concentration

Local analysis

The eigenvalues of the stability matrix connect to those of the strain matrix from which one can see that rotating regions expell (attract) heavy (ligth) particles (Bec JFM 2005)

$$\mathbb{L}_{ij} = \partial_j F_i$$
$$\sigma_{ij} = \partial_j u_i$$

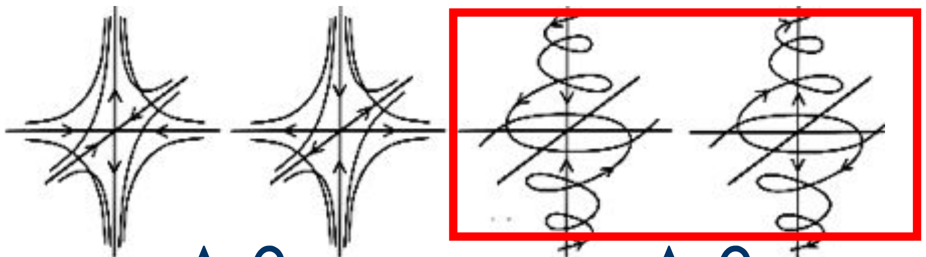
$$\mathbb{L} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \beta D_t \sigma + \frac{\sigma}{St} & -\frac{\mathbb{I}}{St} \end{pmatrix}$$

$$\Delta = \left(\frac{\det[\hat{\sigma}]}{2}\right)^2 - \left(\frac{\mathrm{Tr}[\hat{\sigma}^2]}{6}\right)^3$$

$\Delta \leq 0$ $\quad 3\ \mathcal{R}$ eigen

$\Delta > 0$ $\quad 1\ \mathcal{R} + 2\ \mathcal{C}$ eigen.

Strain regions Rotation regions



$\Delta<0$ $\Delta>0$

d=3 example

Tracers in Incompressible & compressible flows

Thus for St-->0 particles behave approximatively as tracers in compressible flows in dimension d

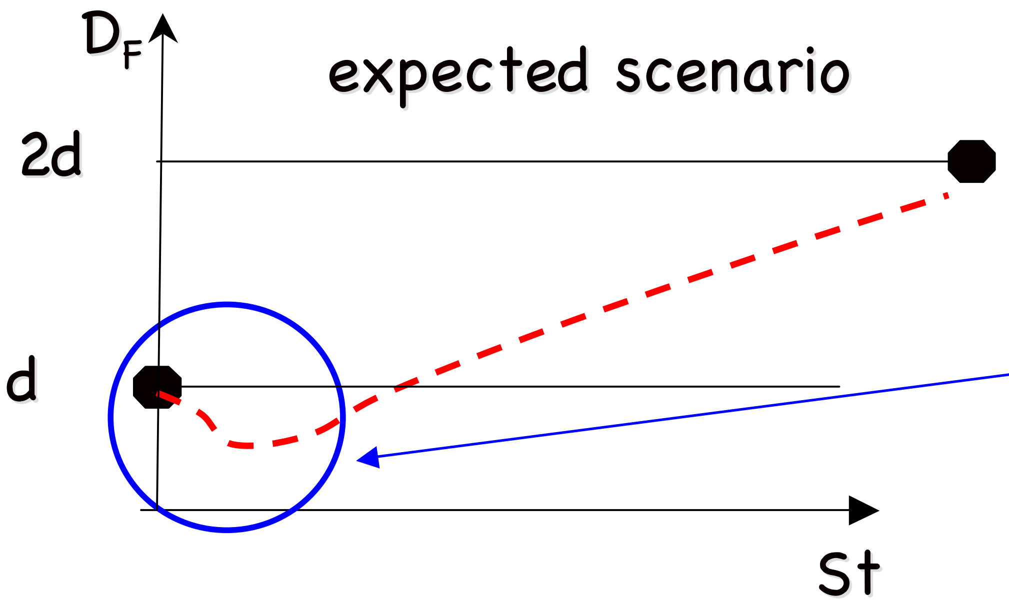
$$\dot{\boldsymbol{X}} = \boldsymbol{V} \approx \boldsymbol{v}(\boldsymbol{X},t) = \boldsymbol{u}(\boldsymbol{X},t) + St(\beta-1)D_t\boldsymbol{u}(\boldsymbol{X},t)$$

$$\dot{\boldsymbol{X}} = \boldsymbol{v}(\boldsymbol{X},t) \qquad \boldsymbol{\nabla} \boldsymbol{v} < 0$$

Dissipative

fractal attractor with

$D_F<d$



$D_F<d$ implies clustering in real space, i.e. the projection of the attractor in real space will be also (multi-)fractal

Clustering in real & phase space

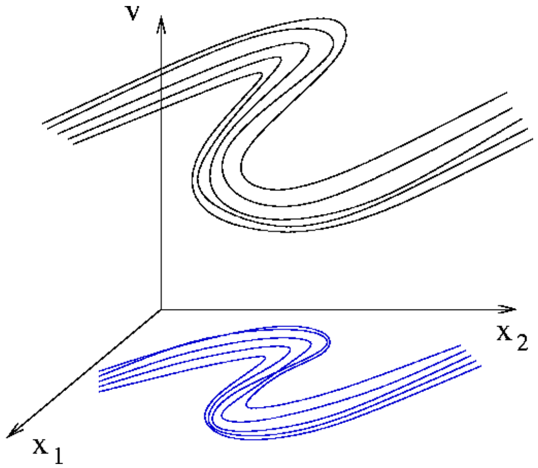
Fractal with $D_F<d$ embedded in a $D=2d$-dimensional (X,V)-phase space, looking at positions only amounts to project it onto a d-dimensional space.

Which will be the observed fractal dimension d_F in position space?

For "isotropic" fractals and "generic" projections

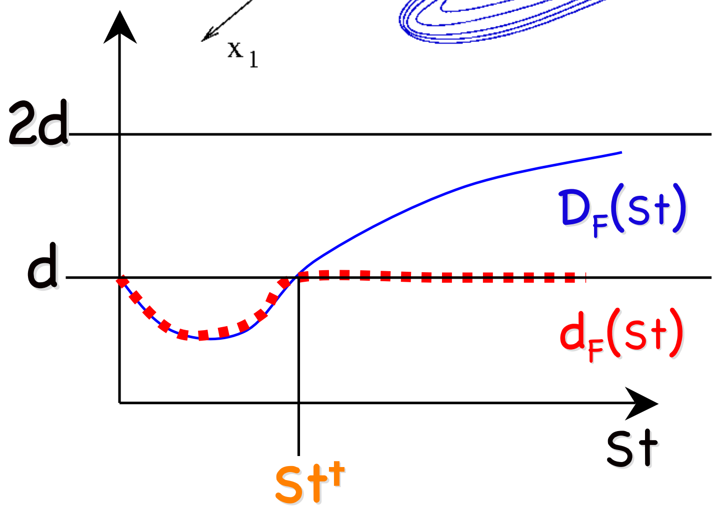
$$d_F = \min\{D_F, d\}$$

(Sauer & Yorke 1997, Hunt & Kaloshin 1997)



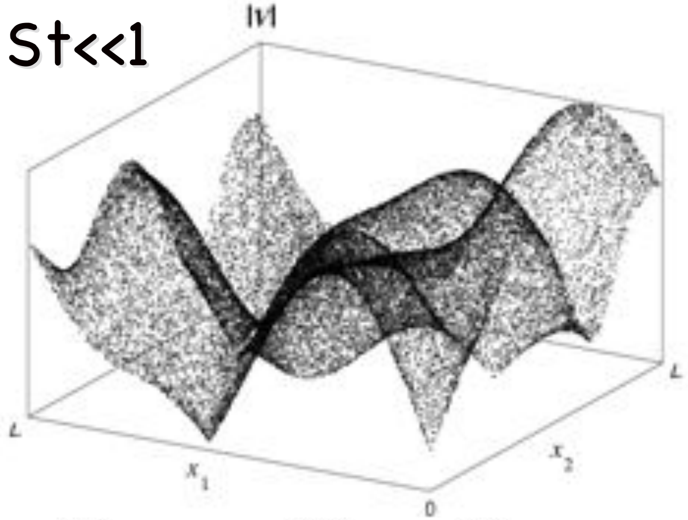
So we expect:

- fractal clustering in physical space with $d_F=D_F$ when $D_F<d$ and $d_F=d$ when $D_F>d$
- existence of critical $St^{\dagger}$ above which no clustering is observed

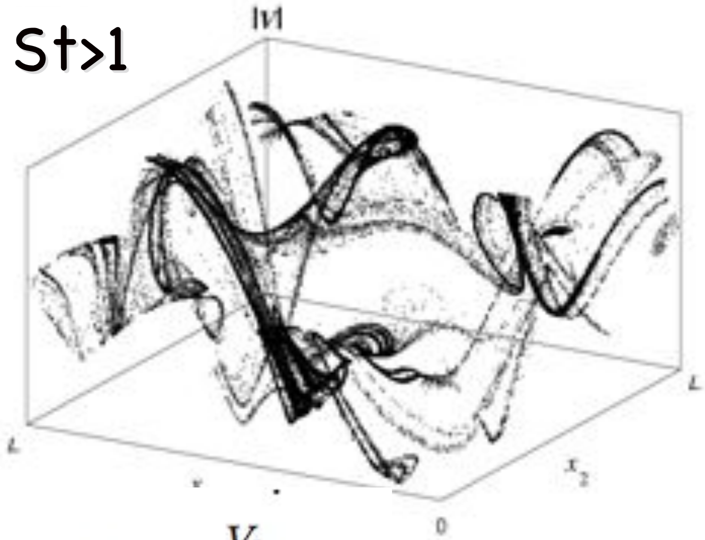


Phase space dynamics

St<<1

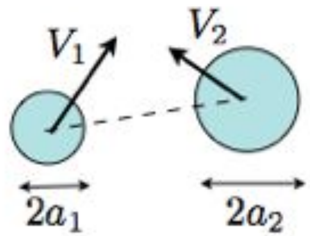


St>1



$$\boldsymbol{R} = \boldsymbol{X}_1 - \boldsymbol{X}_2; \qquad R = |\boldsymbol{R}|$$

$$\delta_R V_{||} = (\boldsymbol{V}_1 - \boldsymbol{V}2) \cdot \frac{\boldsymbol{R}}{R}$$



Collision rate

$$\boxed{k(r) \sim p_2(r)\langle|\delta_R V_{||}|R = r\rangle}$$

$r=a_1+a_2$

Enhanced encounters by clustering

Enhanced relative velocity by caustics

(Falkovich lectures)

Next slides

- Verification of the above picture
  mainly numerical studies, see Toschi lecture for details on the methods

- How generic ?
  comparison between turbulent and simplified flows
  dissipative range physics <-> smooth stochastic velocity fields

- Study of simplified models for systematic numerical and/or analytical investigations
  uncorrelated stochastic velocity fields Kraichnan model
  (Kraichnan 1968, Falkovich, Gawedzki & Vergassola RMP 2001)

Model velocity fields

Time correlated, random, smooth flows:

Ornstein-Uhlenbeck dynamics for a few Fourier modes chosen so to have a statistically homogeneous and isotropic velocity field

$$\frac{d\hat{\boldsymbol{u}}_{\boldsymbol{k}}}{dt} = -\frac{1}{\tau_f}\hat{\boldsymbol{u}}_{\boldsymbol{k}} + c_{\boldsymbol{k}}\boldsymbol{\xi}_{\boldsymbol{k}}\,, \qquad \boldsymbol{u}(\boldsymbol{x},t) = \sum_{k}^{N} \hat{\boldsymbol{u}}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

it can be though as a fair approximation of a Stokesian velocity field

$$\begin{aligned} \partial_t \boldsymbol{u} &= \nu\Delta\boldsymbol{u} + \boldsymbol{f} \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}$$

Advantage

As few modes are considered particles can be evolved without building the whole velocity field, but just computing it where the particles are

Kraichnan model

Gaussian, random velocity with zero mean and correlation

$$\langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t')\rangle = [2\mathfrak{D}_0\delta_{ij} - B_{ij}(\boldsymbol{x}-\boldsymbol{x}')]\boxed{\delta(t-t')}$$

Spatial correlation

$$B_{ij}(\boldsymbol{r}) = \mathfrak{D}_1 r^2[(d+1)\delta_{ij} - 2r_ir_j/r^2]$$ (smooth to mimick dissipative range)

We focus on 2 particle motion allowing for Lagrangian numerical schemes so to avoid to build the whole velocity field

$$\boldsymbol{R} = \boldsymbol{X}_1 - \boldsymbol{X}_2 \qquad \ddot{\boldsymbol{R}} = -\frac{1}{\tau_p}\left(\dot{\boldsymbol{R}} - \delta\boldsymbol{u}(\boldsymbol{R},t)\right)$$

- good approximation for particles with very large Stokes time $\tau_p >> T_L = L/U$ (T_L=integral time scale in turbulence)
- time uncorrelation => no persistent eulerian structures
  only dissipative dynamics is acting (no preferential concentration)
- reduced two particle dynamics amenable of analytical approaches
- can be easily generalized to mimick inertial range physics

$$B_{ij}(\boldsymbol{r}) = \mathfrak{D}_1 r^{2h}[(d-1+2h)\delta_{ij} - 2hr_ir_j/r^2]$$ $0<h<1$ non smooth generalization to mimick inertial range

Kraichnan model

Thanks to time uncorrelation we can write a Fokker-Planck equation for
The joint pdf of separation and velocity difference $p(\boldsymbol{r}, \boldsymbol{v}, t)$

$$\partial_t p + \sum_i \left(\frac{\partial}{\partial r_i} - \frac{1}{\tau_p} \frac{\partial}{\partial v_i} \right) (v_i p) - \frac{1}{\tau_p^2} \sum_{i,j} B_{ij}(\boldsymbol{r}) \frac{\partial^2}{\partial v_i \partial v_j} p = 0$$

$$B_{ij}(\boldsymbol{r}) = \mathfrak{D}_1 r^2 [(d+1)\delta_{ij} - 2r_i r_j / r^2]$$

By rescaling $\begin{cases} t \mapsto t' = t/\tau \\ \boldsymbol{r} \mapsto \boldsymbol{r}' = \boldsymbol{r}/\ell \\ \boldsymbol{v} \mapsto \boldsymbol{v}' = \tau \boldsymbol{v}/\ell \end{cases}$

The statistics only depends on
The Stokes number

$$St = \mathfrak{D}_1 \tau_p$$

Non-smooth generalization

$$B_{ij}(\boldsymbol{r}) = \mathfrak{D}_1 r^{2h} [(d-1+2h)\delta_{ij} - 2h r_i r_j / r^2]$$

$$St(\ell) = \mathfrak{D}_1 \tau_p / \ell^{2(1-h)}$$

$\ell \to \infty \quad \mathrm{St}(\ell) \to 0$ Tracer limit

$\ell \to 0 \quad \mathrm{St}(\ell) \to \infty$ Ballistic limit

Scale dependent
Stokes number
(Falkovich et al 2003)

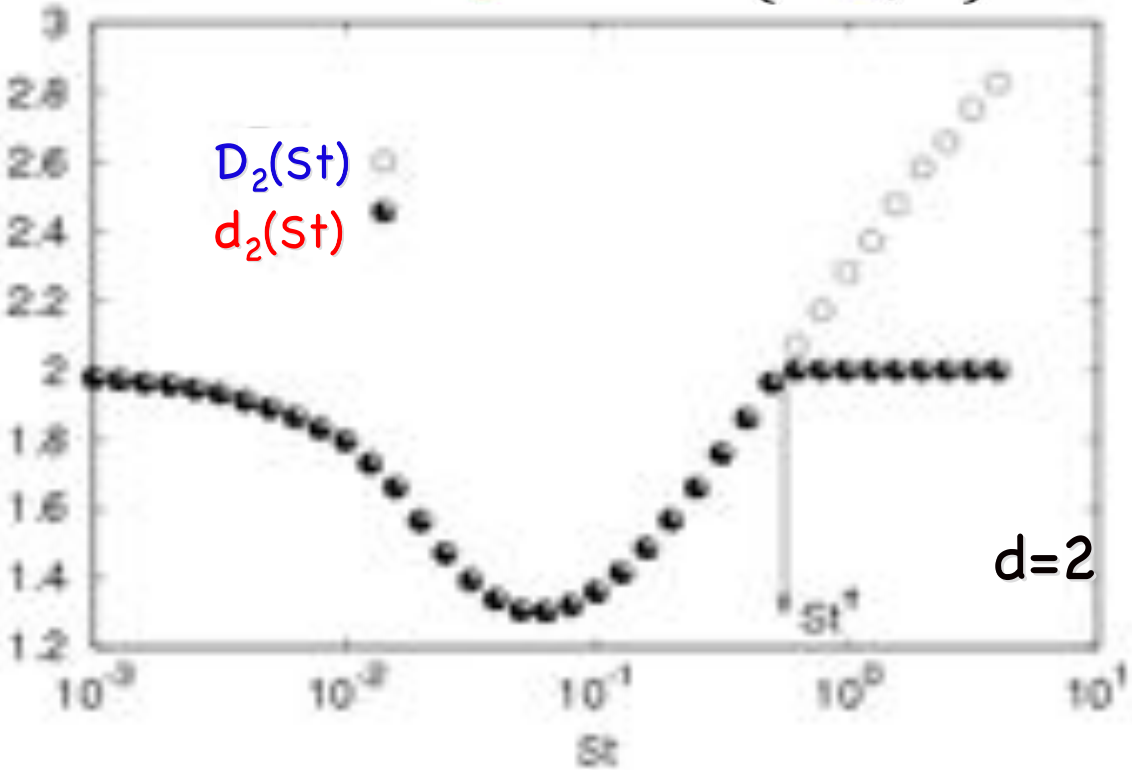
clustering in Kraichnan model

From long time averages of two particles motion

$$P(||\boldsymbol{R}||^2 + ||\dot{\boldsymbol{R}}||^2 < r) \sim r^{D_2}$$ <-Phase-space

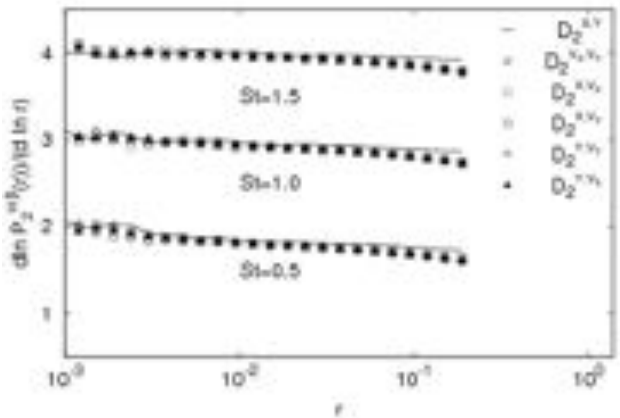
$$P(||\boldsymbol{R}||^2 < r) \sim r^{d_2}$$ <-Position space

$$d_2 = \min\{D_2, d\}$$



Bec, MC, Hillerbrandt & Turitsyn 2008

Different projections X,V_x V_x,V_y... give equivalent results



Evidence of subleading terms, fits must be done with care

$$P_2(r) \approx A r^{d_F} + B r^d$$

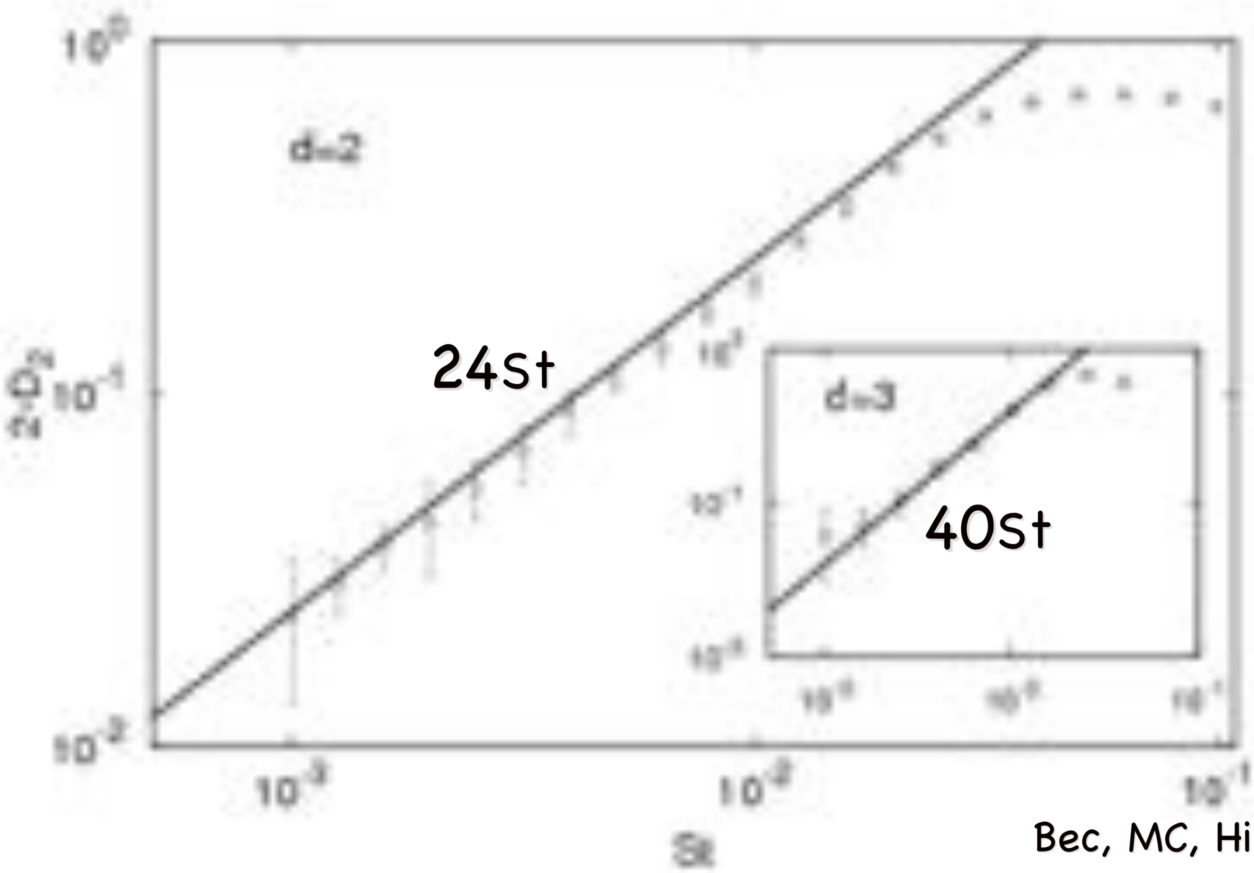
St<<1 Kraichnan

IDEA: for St<<1 velocity dynamics is faster than that of the separation

Stochastic averaging method

(Majda, Timofeyev & Vanden Eijnden 2001)

$$p(\boldsymbol{r},\boldsymbol{v}) = p(\boldsymbol{r})P_{\boldsymbol{r}}(\boldsymbol{v}) + \text{h.o.t}$$



- Stationary solution for the velocity
- Perturbative Expansion in the slow variable (the separation)

Deviation from d is linear in St

$$\mathcal{D}_2 = d - 2(d+1)(d+2)\,\mathrm{St} + \mathrm{O}(\mathrm{St}^2)$$

Bec, MC, Hillerbrand & Turitsyn, (2008)

Results agree with

Wilkinson, Mehlig & Gustavsson (2010)

and Olla (2010)

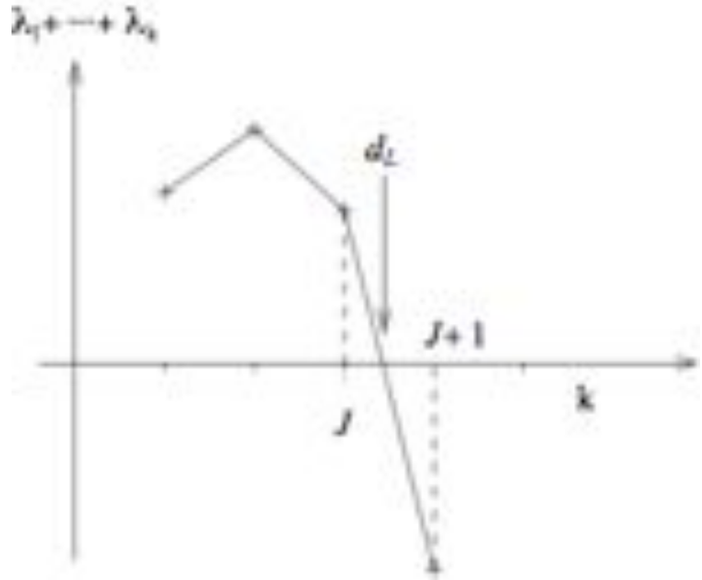
Clustering in random smooth flows
(time correlated)

$$\begin{aligned}\dot{\boldsymbol{X}} &= \boldsymbol{V}\\ \dot{\boldsymbol{V}} &= \beta D_t \boldsymbol{u}(\boldsymbol{X}) + \frac{\boldsymbol{u}(\boldsymbol{X},t)-\boldsymbol{V}}{St}\end{aligned}$$

$$\frac{d\hat{\boldsymbol{u}}_{\boldsymbol{k}}}{dt} = -\frac{1}{\tau_f}\hat{\boldsymbol{u}}_{\boldsymbol{k}} + c_{\boldsymbol{k}}\boldsymbol{\xi}_{\boldsymbol{k}}\,, \qquad \boldsymbol{u}(\boldsymbol{x},t) = \sum_{k}^{N}\hat{\boldsymbol{u}}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

(Bec 2004,2005)

We can estimate the dimension on the attractor in terms of
The Lyapunov dimension



$$D_L = J + \frac{\sum_{i=1}^{J}\lambda_i}{|\lambda_{J+1}|}$$

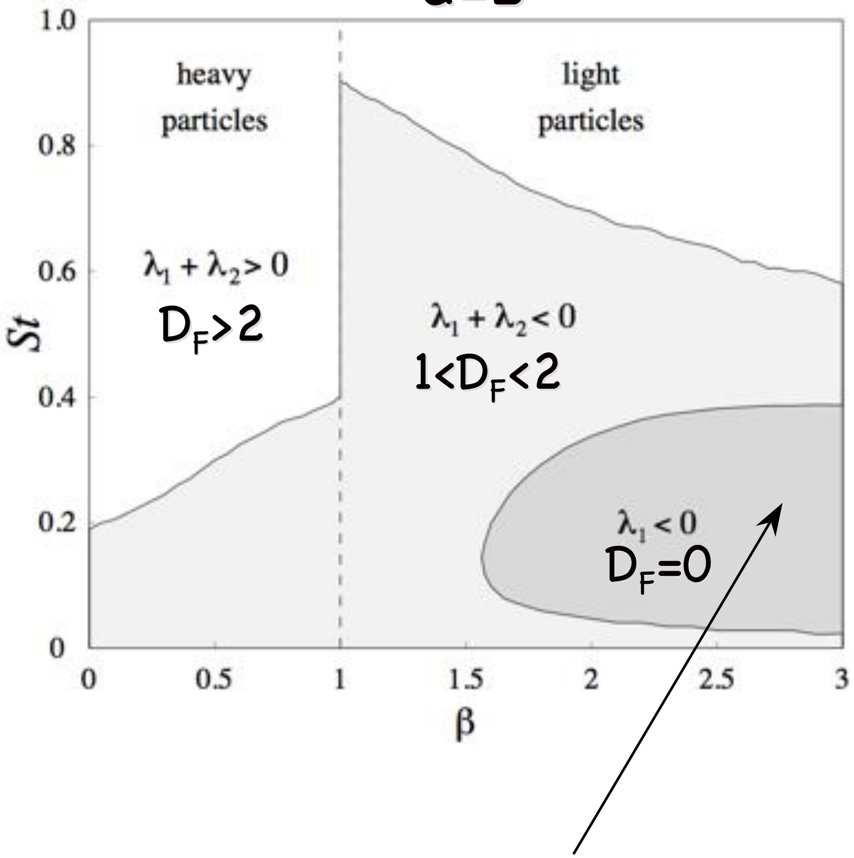
Conditions for D_L=integer

$\lambda_1=0$	$D_L=1$
$\lambda_1+\lambda_2=0$	$D_L=2$
$\lambda_1+\lambda_2+\lambda_3=0$	$D_L=3$

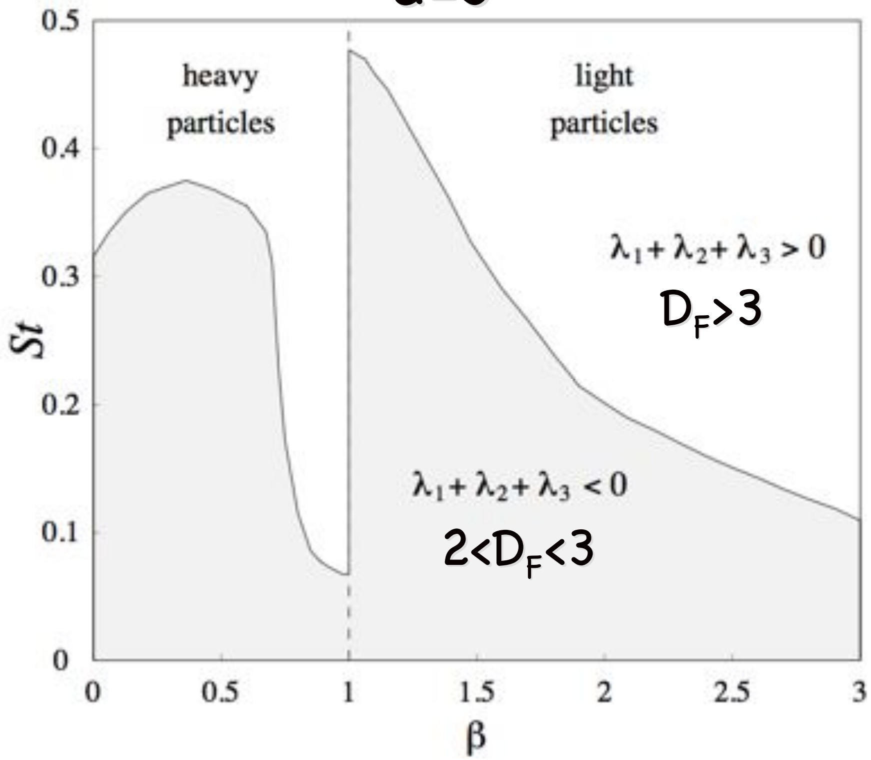
Looking at the first, sum of first 2 or sum of first 3 Lyapunov exponents we can have a picture of the (β,St) dependence of the fractal dimension

(β,St)-phase diagram

d=2



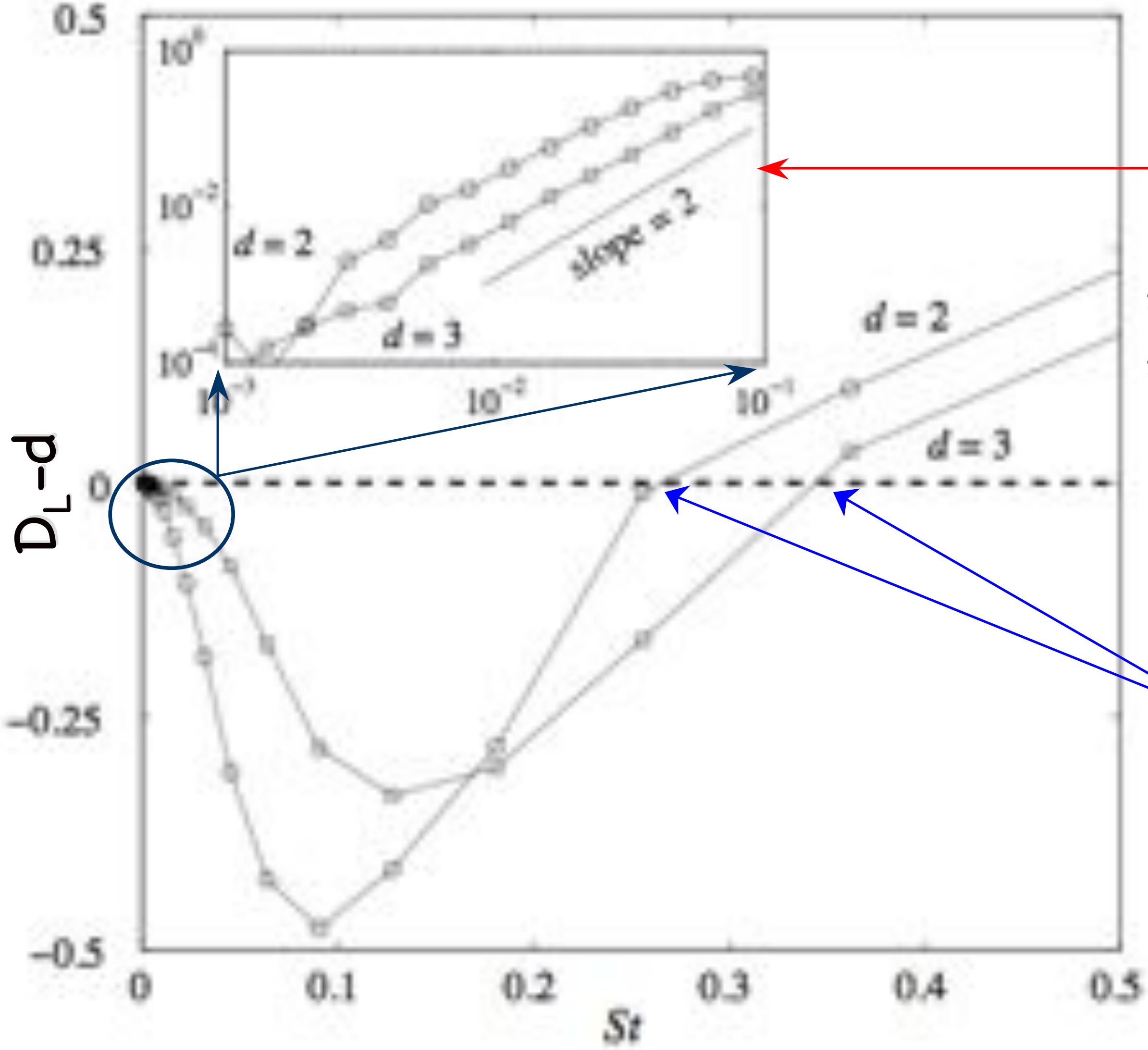
d=3



Light Particles being attracted in point-like attractors (trapping in vortices)

Notice that $D_F>2$ always vortical structure Seems to be not effective in trapping Ligth particles

Lyapunov dimension for β=0



Deviation from d
is quadratic in St

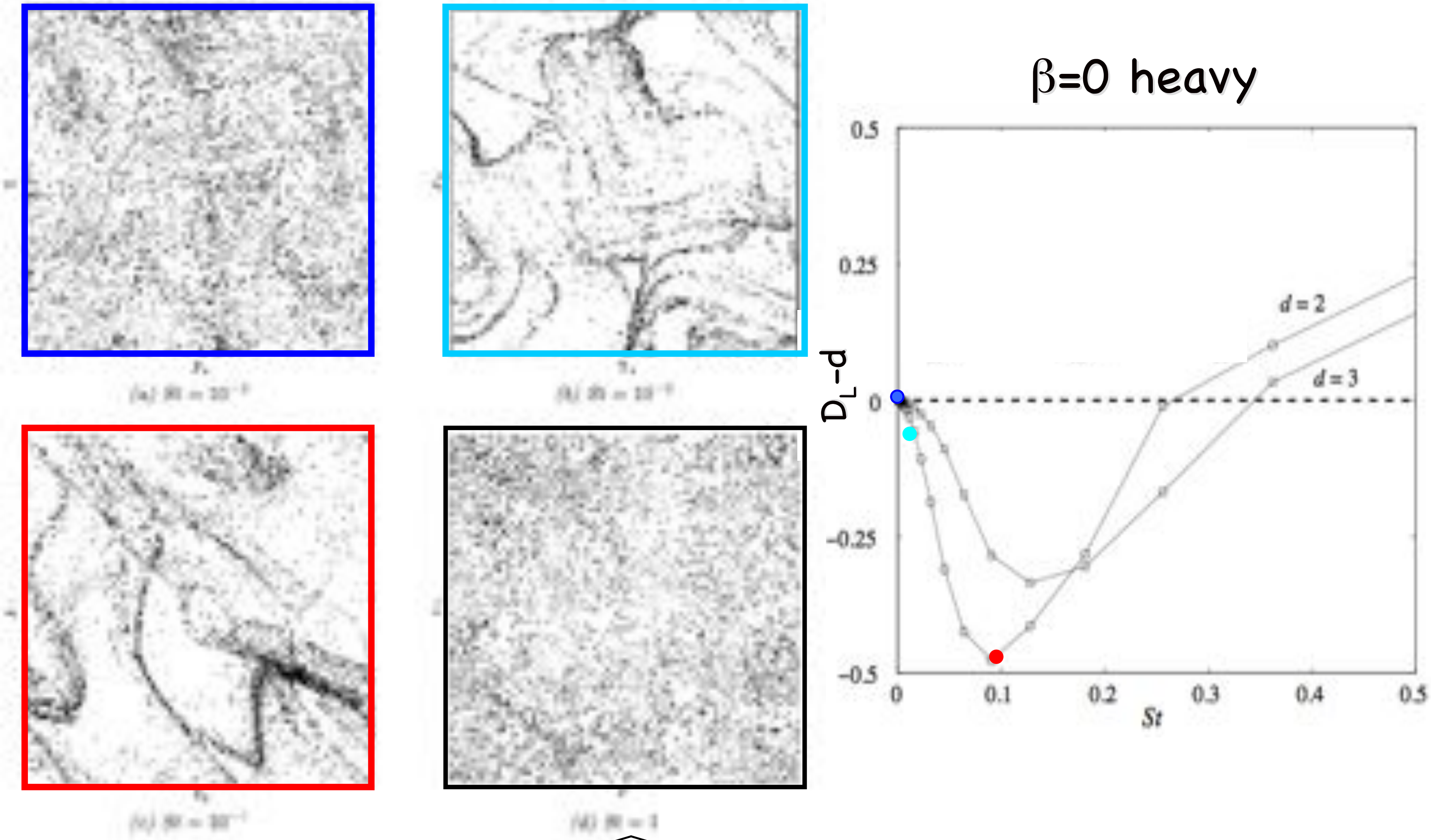
$$D_L(St) \approx d - \alpha St^2$$

in uncorrelated flows
is linear

$St^{\dagger}$

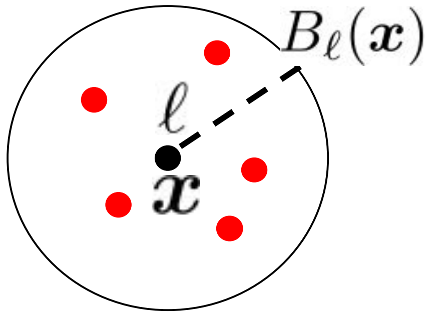
Critical St for clustering
in position space

Clustering in position space

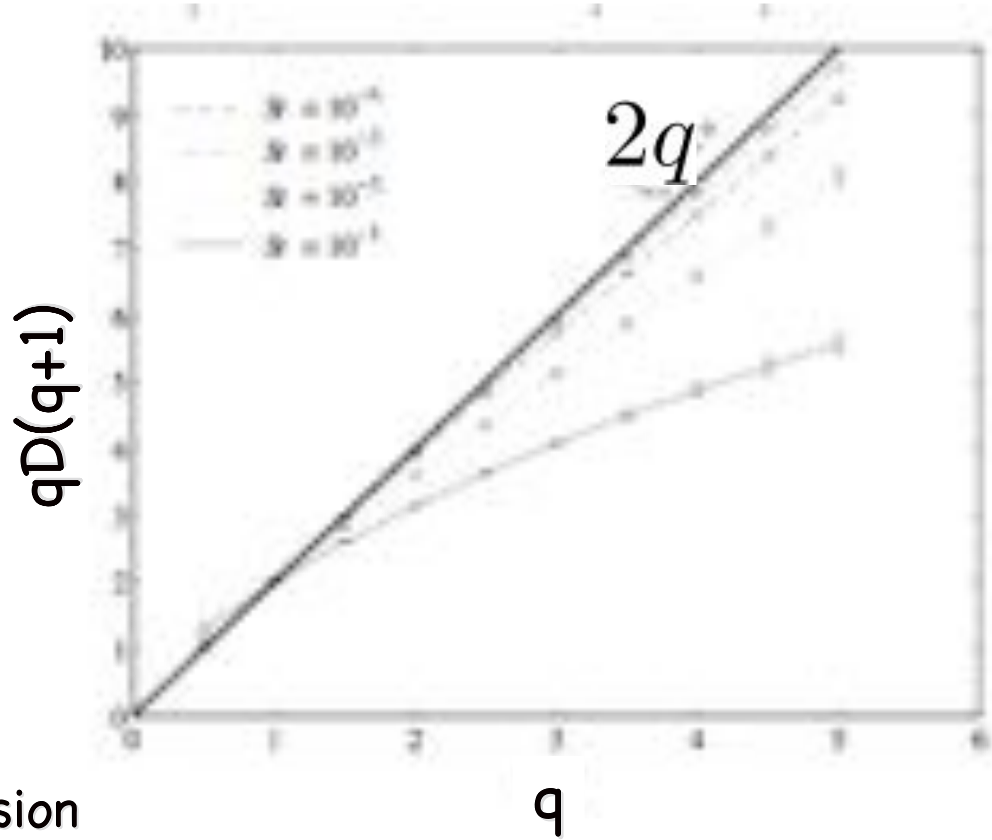


$\boxed{St > St^{\dagger}}$ No clustering

Multifractality



$$\boxed{\langle [p_{B_\ell(\boldsymbol{x})}]^q \rangle \sim \ell^{qD(q+1)}}$$



$D(0) = D_F$ Fractal dimension

$D(1) = \lim_{\ell \to 0} \frac{\sum_{n=0}^{N(\ell)} p_n(\ell) \ln p_n(\ell)}{\ln \ell}$ Information dimension

$D(2) = D_{corr}$ Correlation dimension $P_2(||\boldsymbol{x}_1 - \boldsymbol{x}_2|| < r) \sim r^{D(2)}$

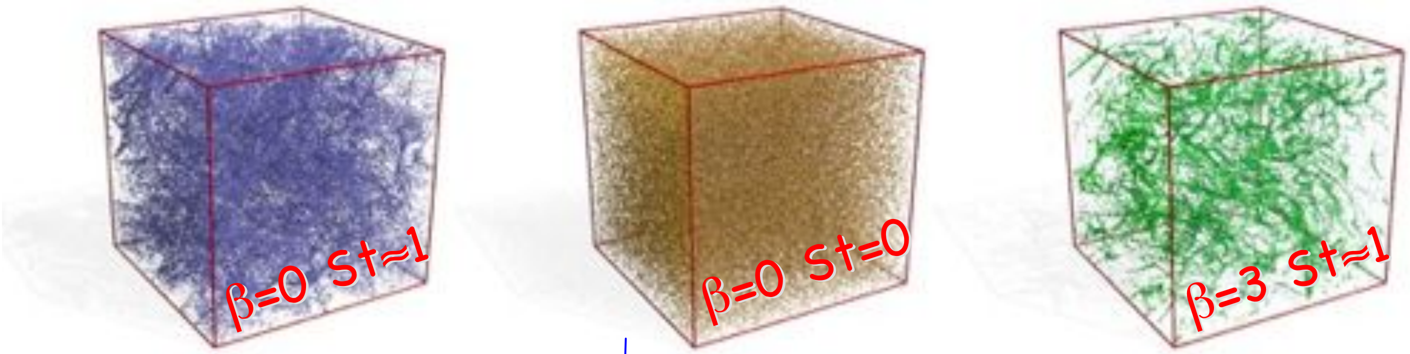
$D(n)$ n integer: controls the probability to find n particles in a ball of size r

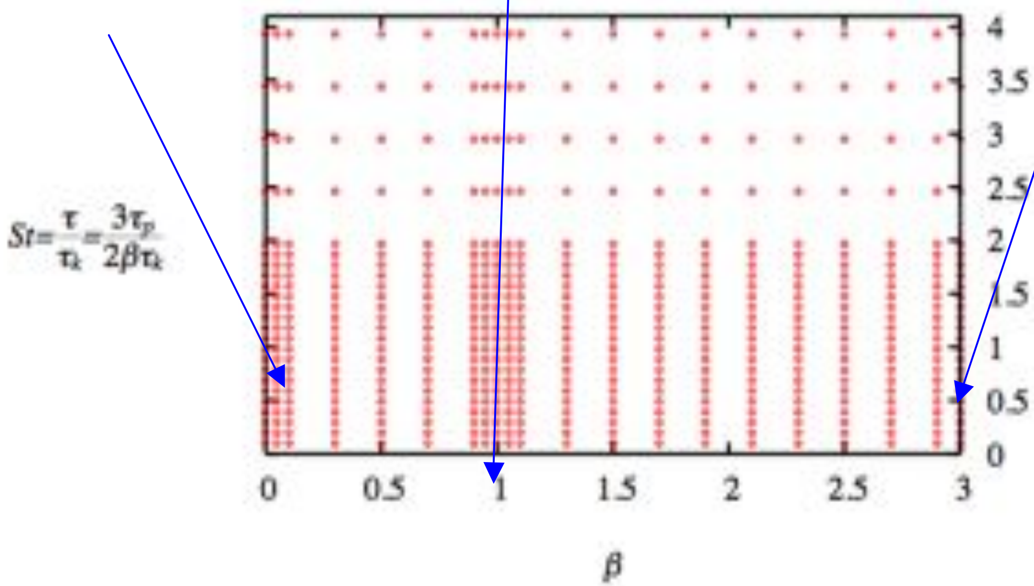
Particles in turbulence

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \tfrac{1}{\rho_f}\boldsymbol{\nabla}p + \boldsymbol{f}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

$$\begin{aligned}\dot{\boldsymbol{X}} &= \boldsymbol{V} \\ \dot{\boldsymbol{V}} &= \beta D_t \boldsymbol{u}(\boldsymbol{X}) + \frac{\boldsymbol{u}(\boldsymbol{X},t) - \boldsymbol{V}}{St}\end{aligned}$$



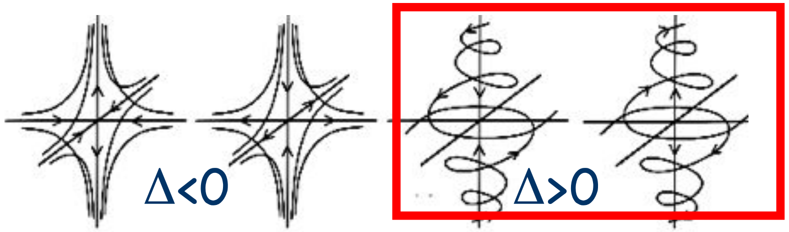


DNS summary

N^3	Re_λ	β	St range
512^3	185	0->3	0.16->4
128^3	65	0->3	0.16->4
2048^3	400	0	0.16->70
512^3	185	0	0.16->3.5
256^3	105	0	0.16->3.5
128^3	65	0	0.16->3.5

Preferential concentration

Strain rotation



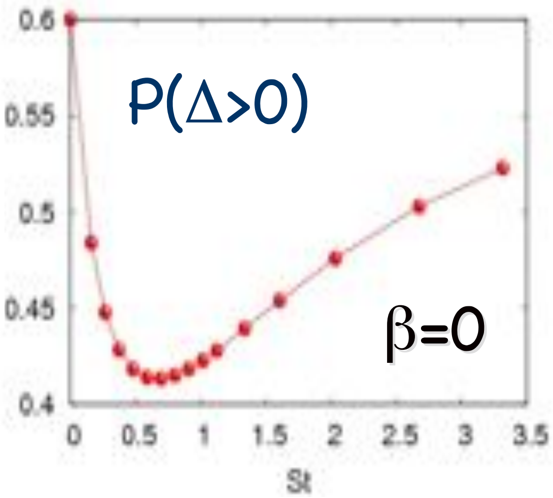
$$\Delta = \left(\frac{\det[\hat{\sigma}]}{2}\right)^2 - \left(\frac{\mathrm{Tr}[\hat{\sigma}^2]}{6}\right)^3 \qquad \hat{\sigma}_{ij} = \partial_i u_j$$

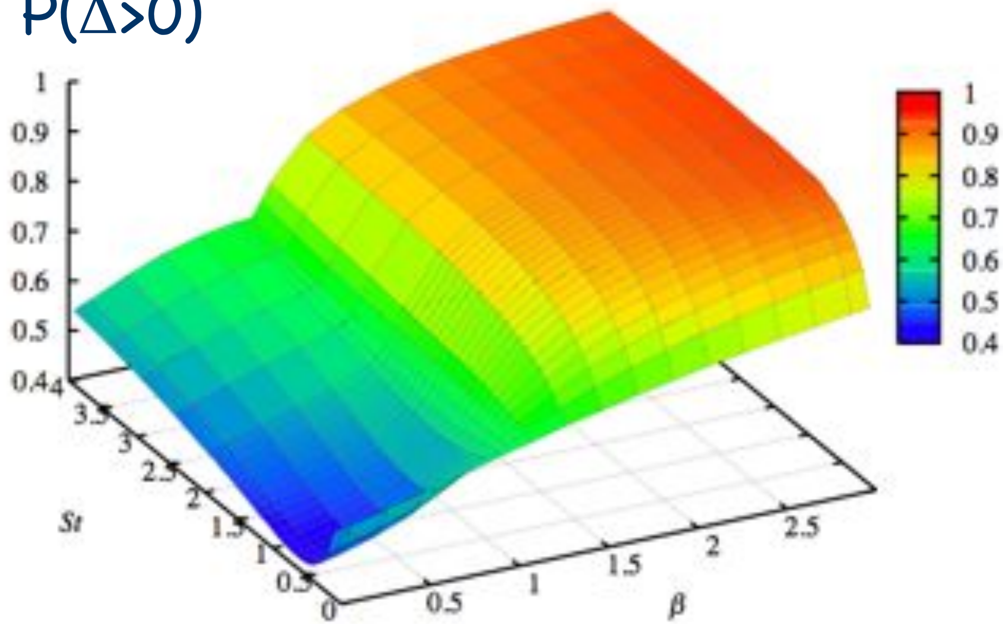
$$\Delta \leq 0 \qquad 3\ \mathcal{R}\ \text{eigen}$$

$$\Delta > 0 \quad 1\ \mathcal{R} + 2\ \mathcal{C}\ \text{eigen.}$$

Heavy particles like strain regions

Light particles like rotating regions

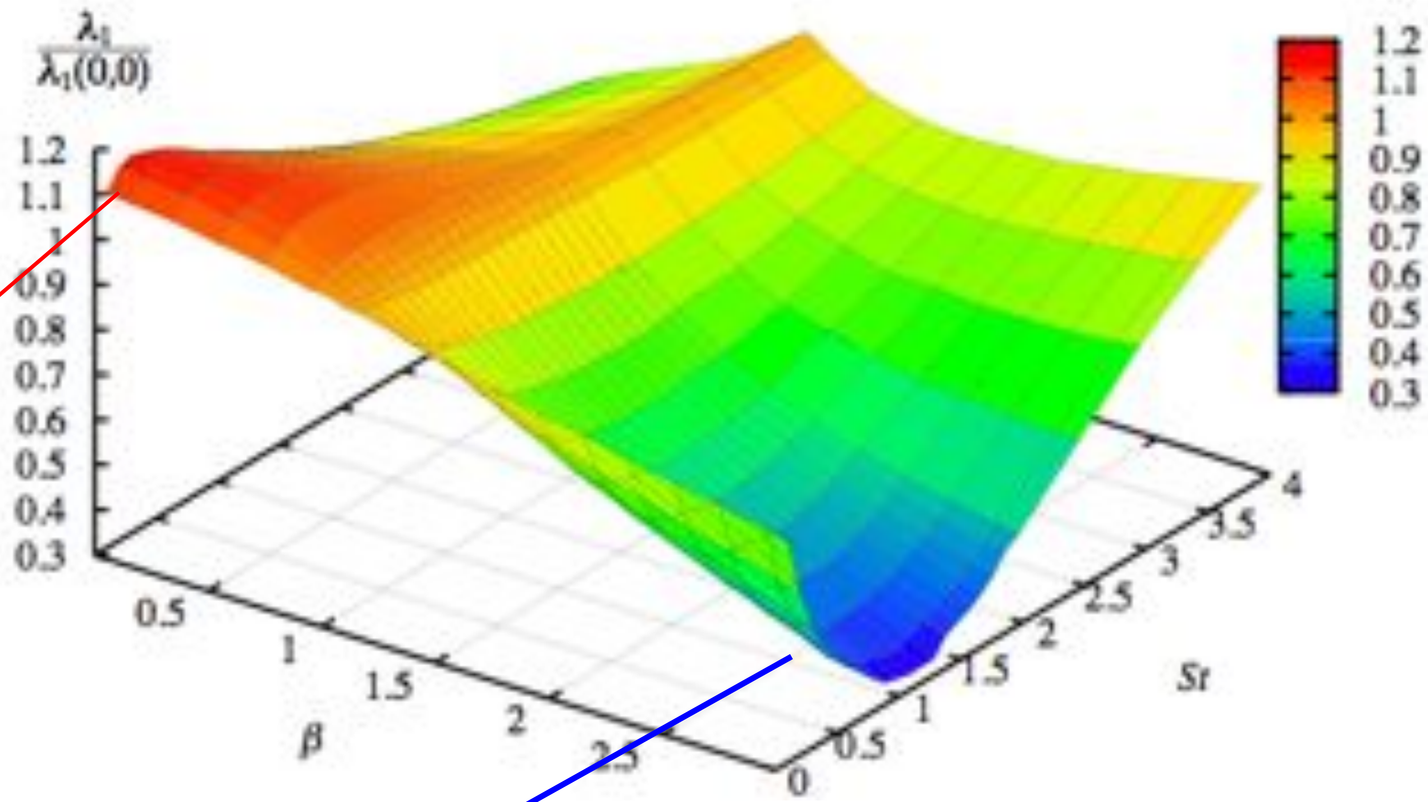




Correlations with the flow are stronger for light particles

Bec et al (2006)

Lyapunov exponent



Heavy
St<<1
$\lambda_1(St) > \lambda_1(St=0)$

stay longer in strain-regions

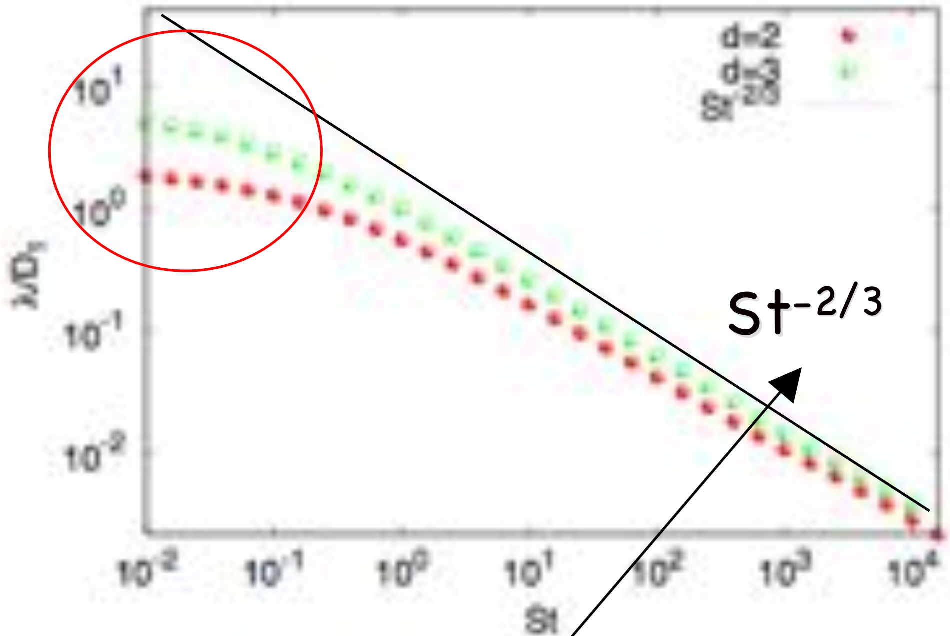
Due to
uneven distribution of particles

Light
$\lambda_1(St) < \lambda_1(St=0)$
staying away from strain-regions

Calzavarini, MC, Lohse & Toschi 2008

Lyapunov exponents

This effect is absent in uncorrelated Flows (Kraichnan), absence of persistent Eulerian tructures:
preferential concentration is not effective
Actually in this case PC should be understood as a cumulative effect on the particle history (P. Olla 2010)

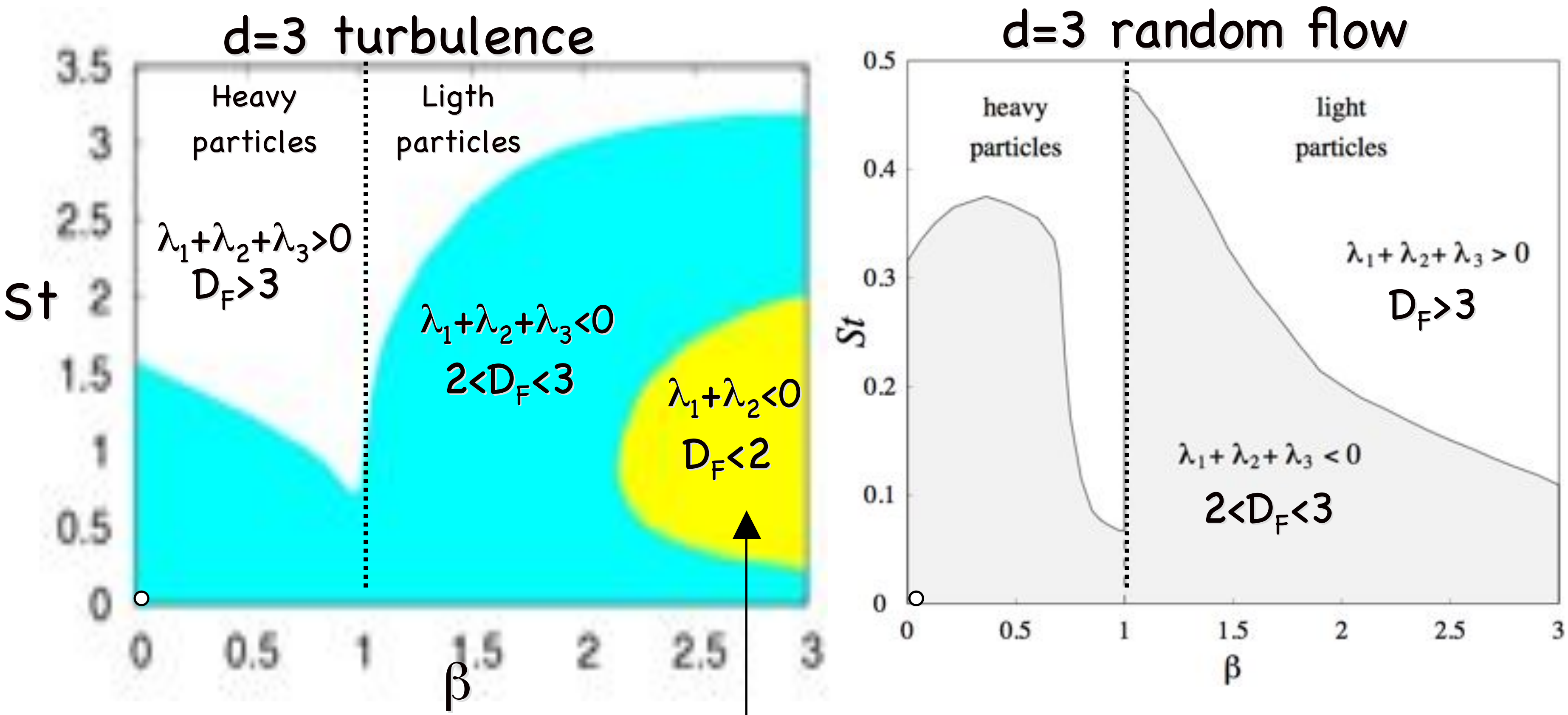


$St^{-2/3}$

The effect can be analytically studied systematically in correlated stochastic flows with telegraph noise
(Falkovich, Musacchio,Piterbarg & Vucelja (2007)

Large St asymptotics
Valid also in correlated flows
Expected in turbulence for $\tau_p >> T_L$

(β,St)-phase diagram

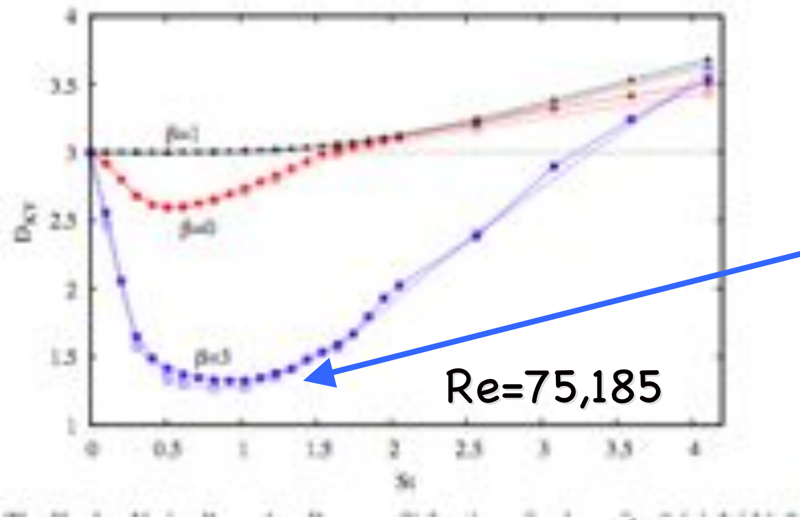


Signature of vortex filaments?

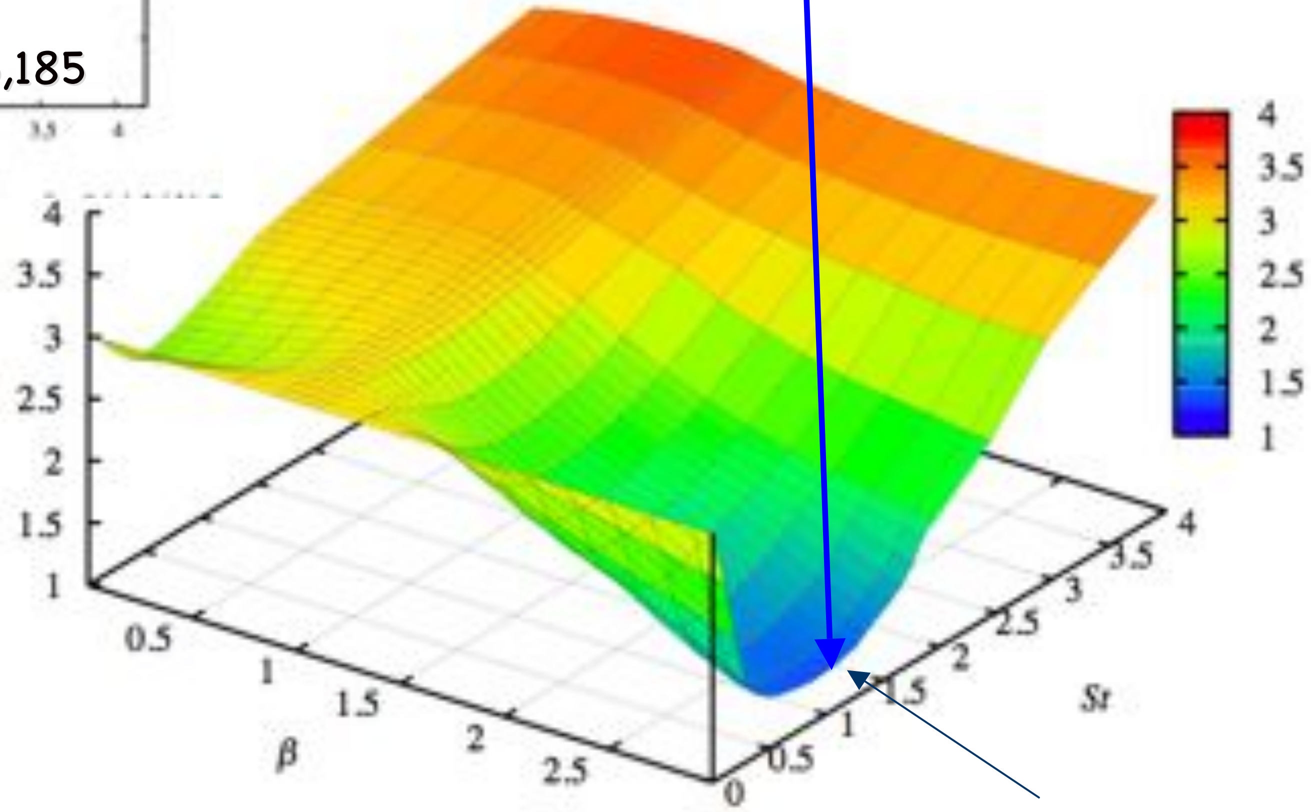
Which are known to be long-lived in turbulence

Lyapunov Dimension

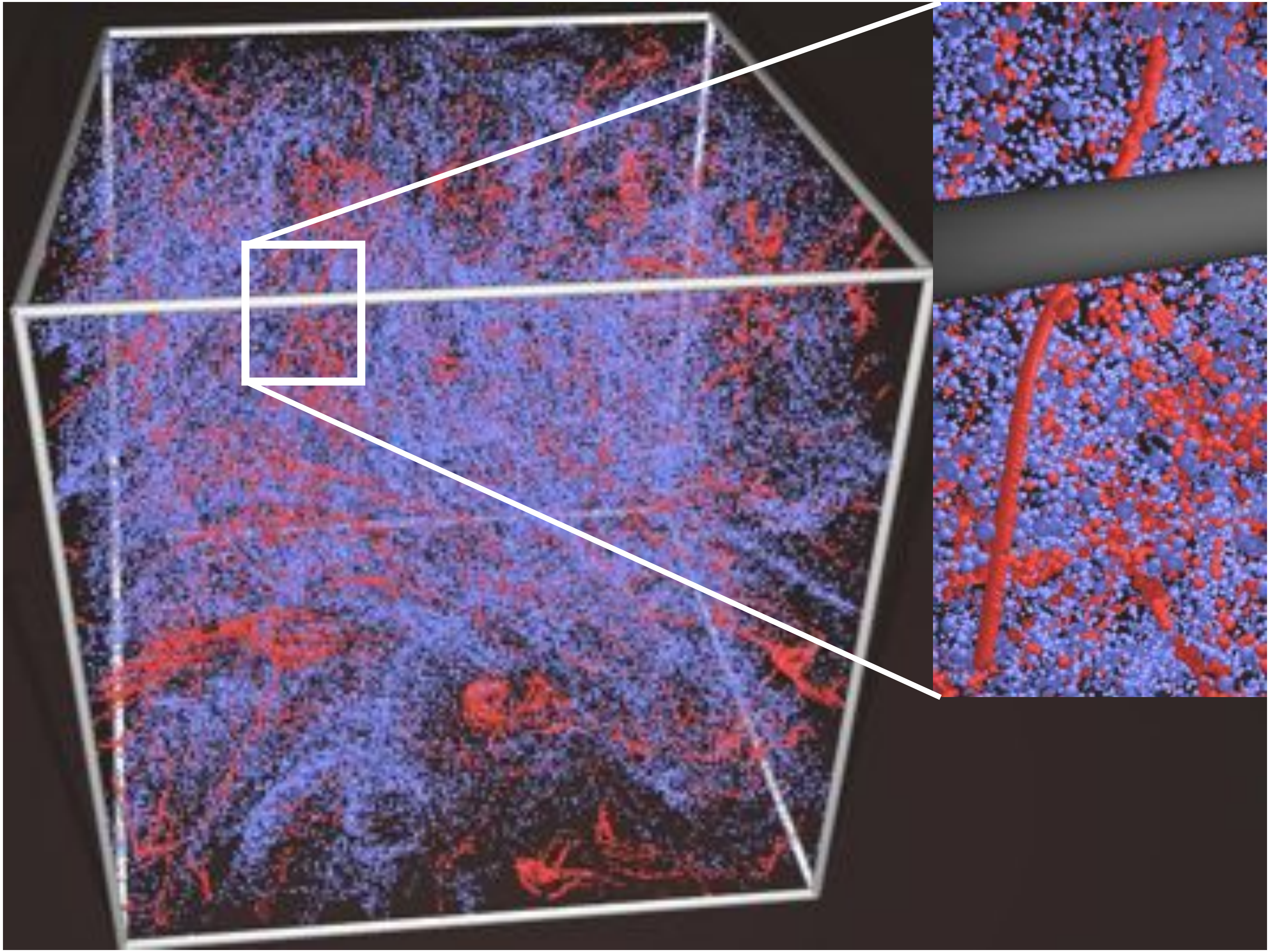
Light particles stronger clustering
$D_2 \approx 1$ signature of vortex filaments



$$d_\lambda = \mathrm{K} + \frac{\sum_{i=1,\mathrm{K}}^{+} \lambda_i}{|\lambda_{\mathrm{K}+1}|}$$

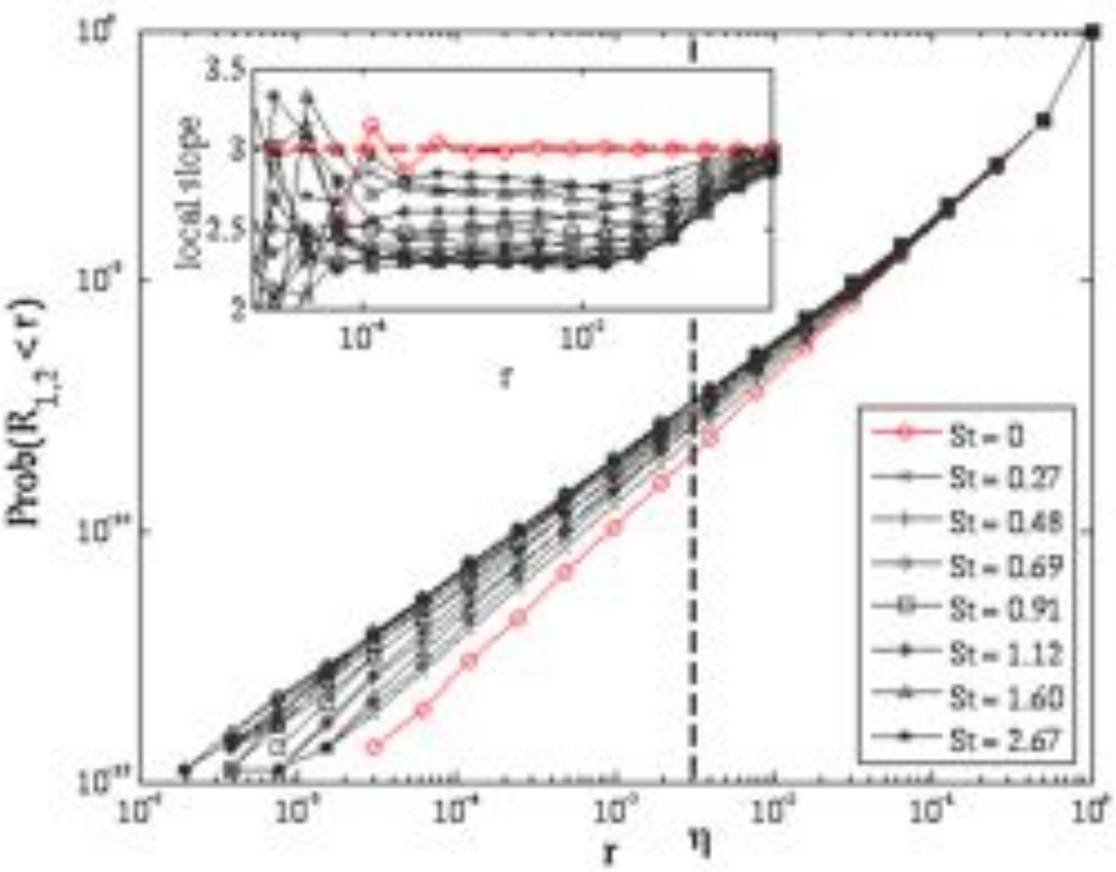


Light particles: neglecting collisions might be a problem!



Clustering of heavy particles in position space

- Dissipative range -->Smooth flow -> fractal distribution
- Everything must be a function of **St_η & Re_λ** only (β=0)



correlation dimension

$$P_2^<(r) \sim r^{\mathcal{D}_2}$$

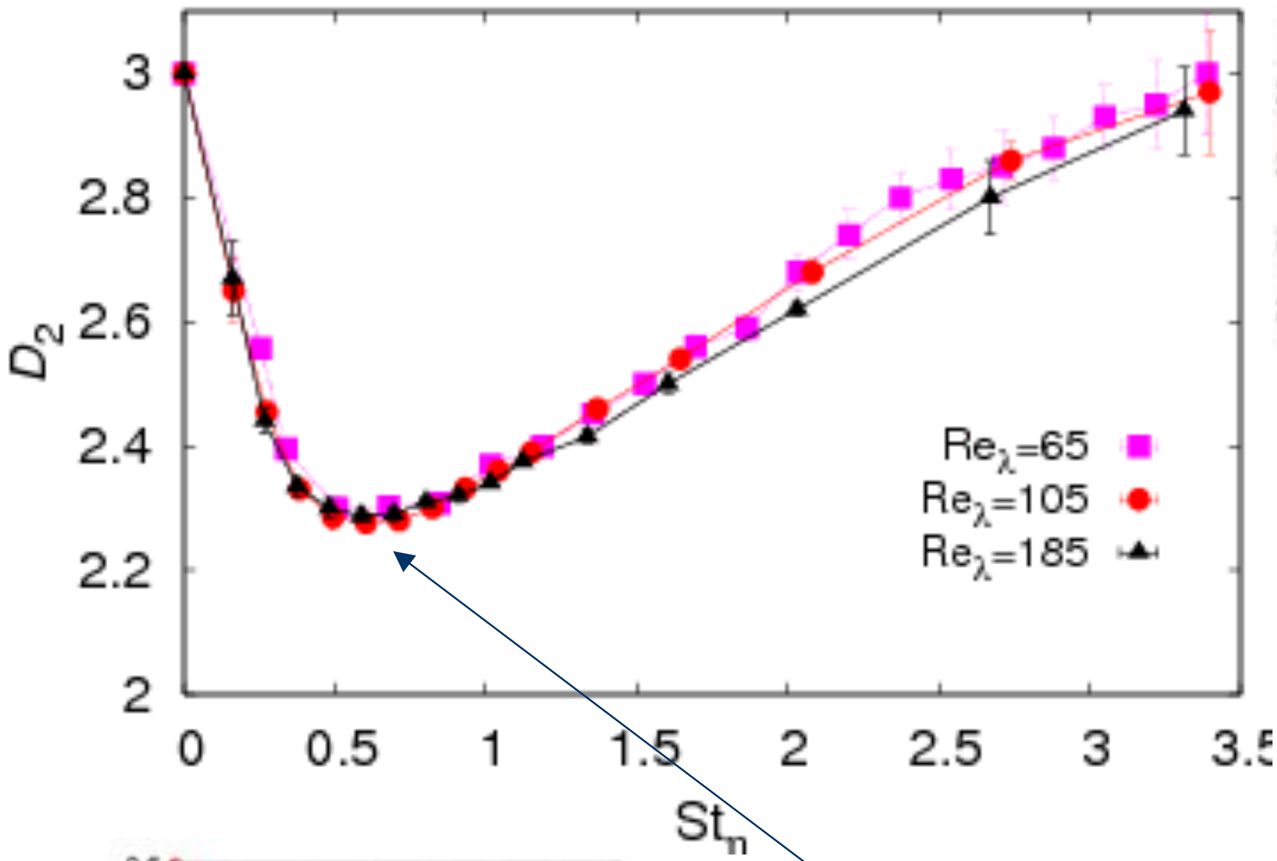
Related to radial distribution function

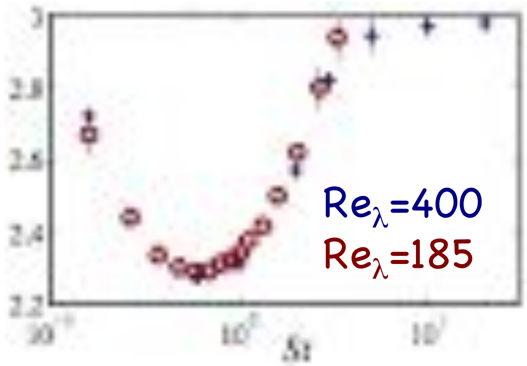
Sundaram & Collins (1997)

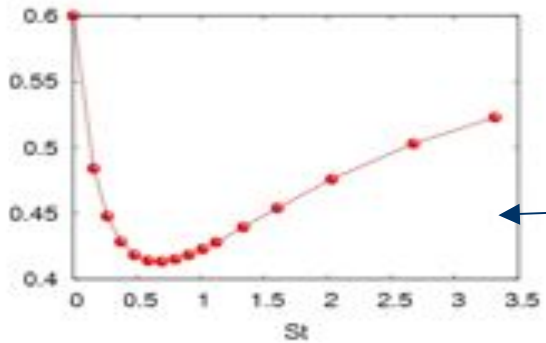
Zhou, Wexler & Wang (2001)

$$g(r) \propto r^{\mathcal{D}_2 - d} \quad \mathcal{D}_2 - d < 0$$

Correlation Dimension $(\beta=0)$

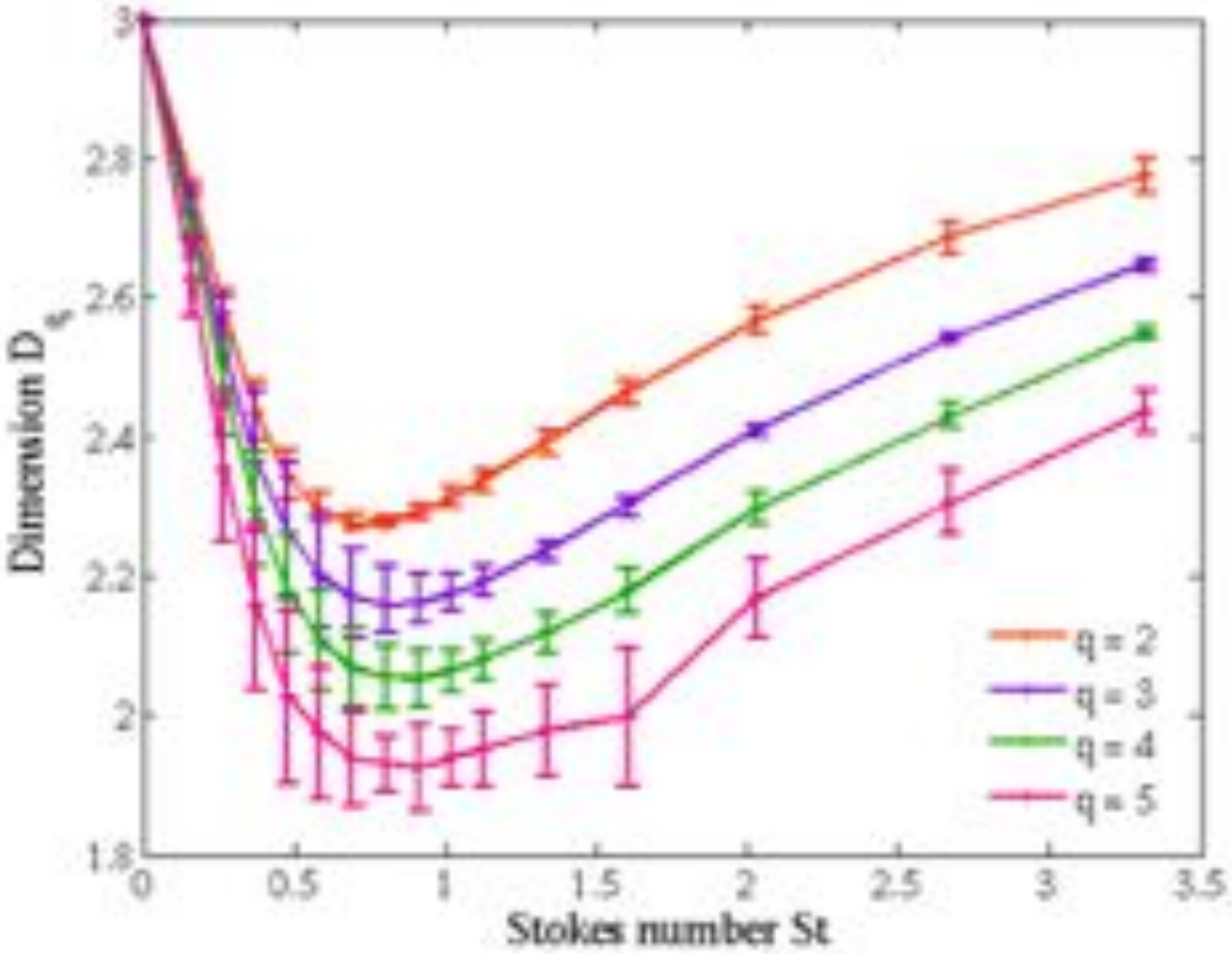






- Maximum of clustering for $St_\eta \approx 1$
- D_2 almost independent of Re_λ
- Link between clustering and Preferential concentration

Multifractality



$D(q) \neq D(0)$

Briefly other aspects

- How to treat polydisperse suspensions?
  - Can we extend the treatment to suspensions of particles having different density or size (Stokes number)? Important for heuristic model of collisions

  (for details see Bec, Celani, MC, Musacchio 2005)

- What does happen at inertial scales?
  - So far we focused on clustering at very small scales (in the dissipative range $r<\eta$) what does happen while going at inertial scales ($\eta<<r<<L$)?

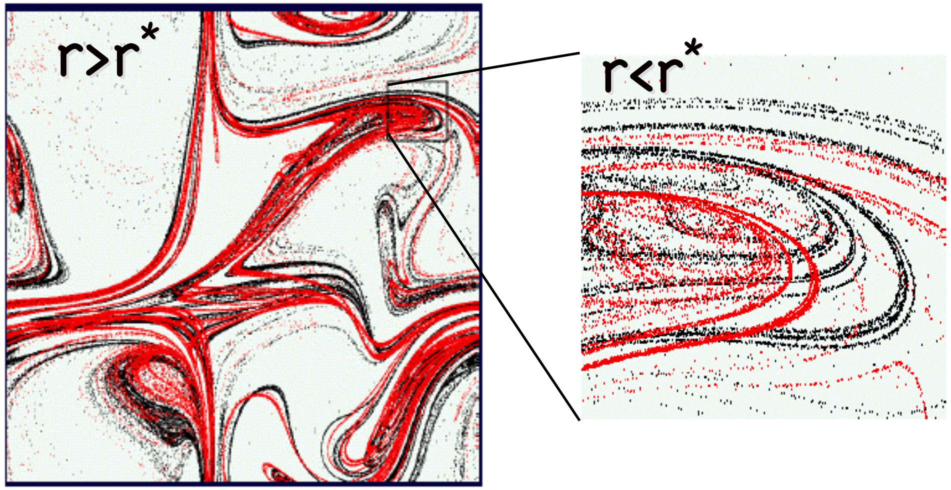
  (for details see Bec, Biferale, MC, Lanotte, Musacchio & Toschi 2007

  Bec, MC. & Hillerbrandt 2007; Bec, MC, Hillerbrandt & Turitsyn 2008)

Polydisperse suspensions

e.g. β=0 with St_1 and St_2

- St_1=St_2 same attractor
- $St_1 \approx St_2$ "close attractors"
- there is a length scale $r_* = \eta \left| \frac{\Delta St}{St} \right|$



$$\boldsymbol{R} = \boldsymbol{X}^{(1)} - \boldsymbol{X}^{(2)} \quad \boldsymbol{W} = \boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}$$

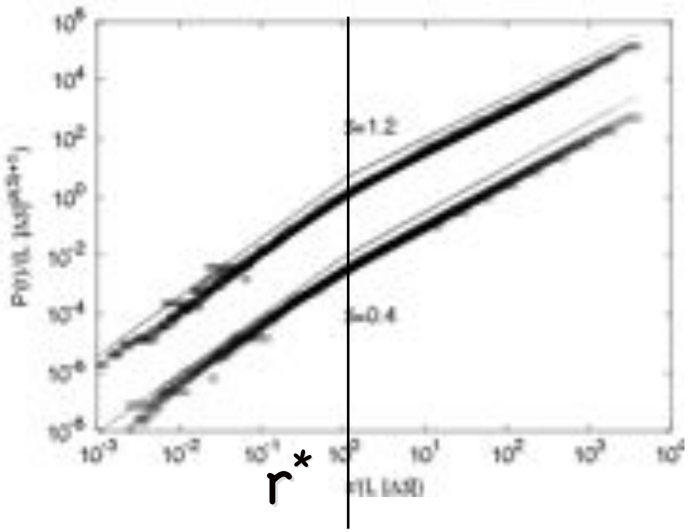
$$\Delta \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{X}^{(1)}) - \boldsymbol{u}(\boldsymbol{X}^{(2)}) \propto R$$

$$\frac{d\boldsymbol{R}}{dt} = \boldsymbol{W}\,, \qquad \frac{d\boldsymbol{W}}{dt} = \frac{1}{\tau} \frac{\Delta \boldsymbol{u} - \boldsymbol{W}}{1 - (\frac{\Delta St}{4\bar{S}t})^2} - \frac{1}{\tau} \frac{\Delta St}{St} \frac{\bar{\boldsymbol{u}} - \bar{\boldsymbol{V}}}{1 - (\frac{\Delta St}{4\bar{S}t})^2}$$

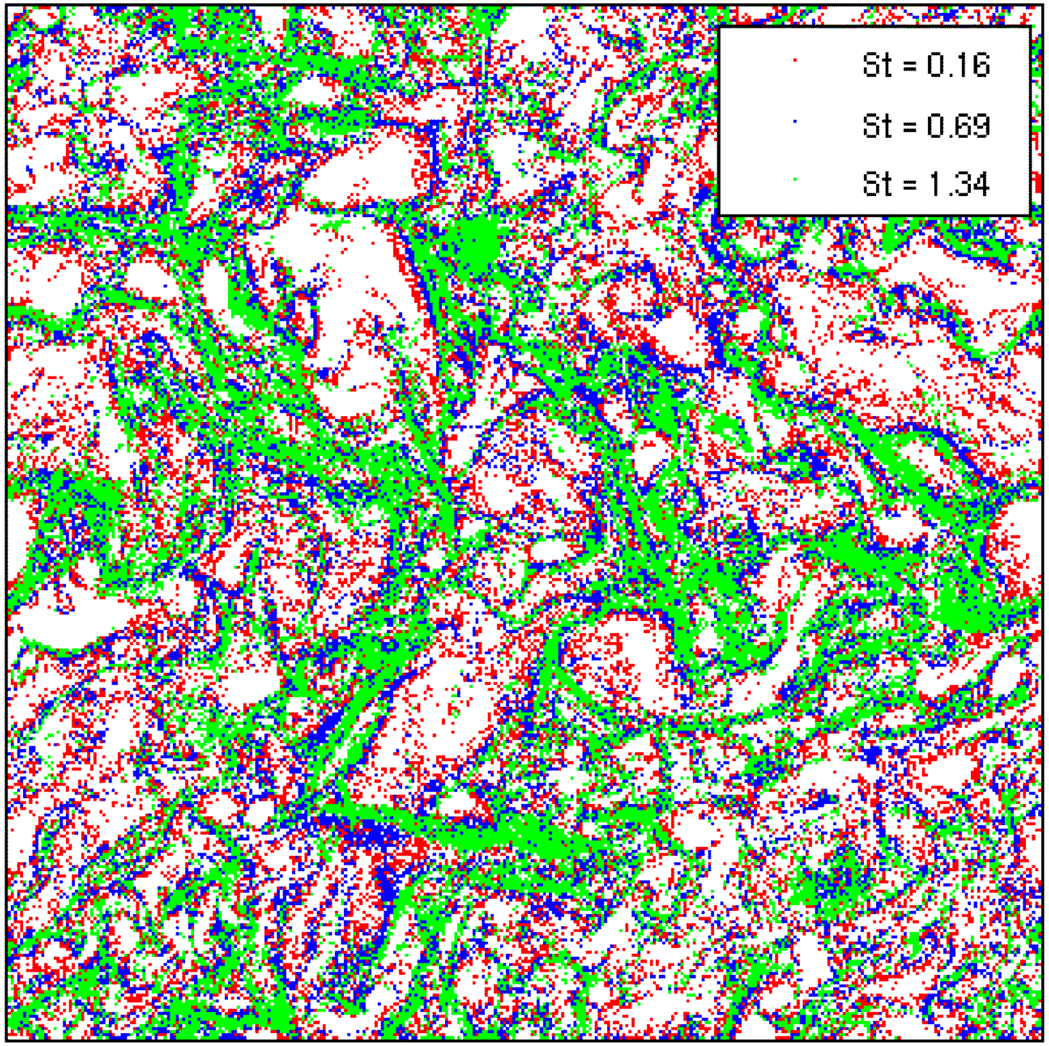
$$\bar{S}t = (St_1 + St_2)/2$$

$$P_2(r) \sim \begin{cases} r^d & r < r^* \quad \text{<-uncorrelated} \\ r^{d_2(\bar{S}t)} & r > r^* \quad \text{<-correlated (through the fluid)} \end{cases}$$

Relevant to collisions between particles with different Stokes



What does happen in the inertial range?



- Voids & structures from η to L
- Distribution of particles over scales?
- What is the dependence on St_η? Or what is the proper parameter?

Insights from Kraichnan model

$$B_{ij}(\boldsymbol{r}) = \mathfrak{D}_1 r^{2h}[(d-1+2h)\delta_{ij} - 2h r_i r_j / r^2]$$

h=1 dissipative range

h<1 inertial range

The statistics only depends on the local Stokes number

$$St(\ell) = \mathfrak{D}_1 \tau_p / \ell^{2(1-h)}$$

Tracer limit

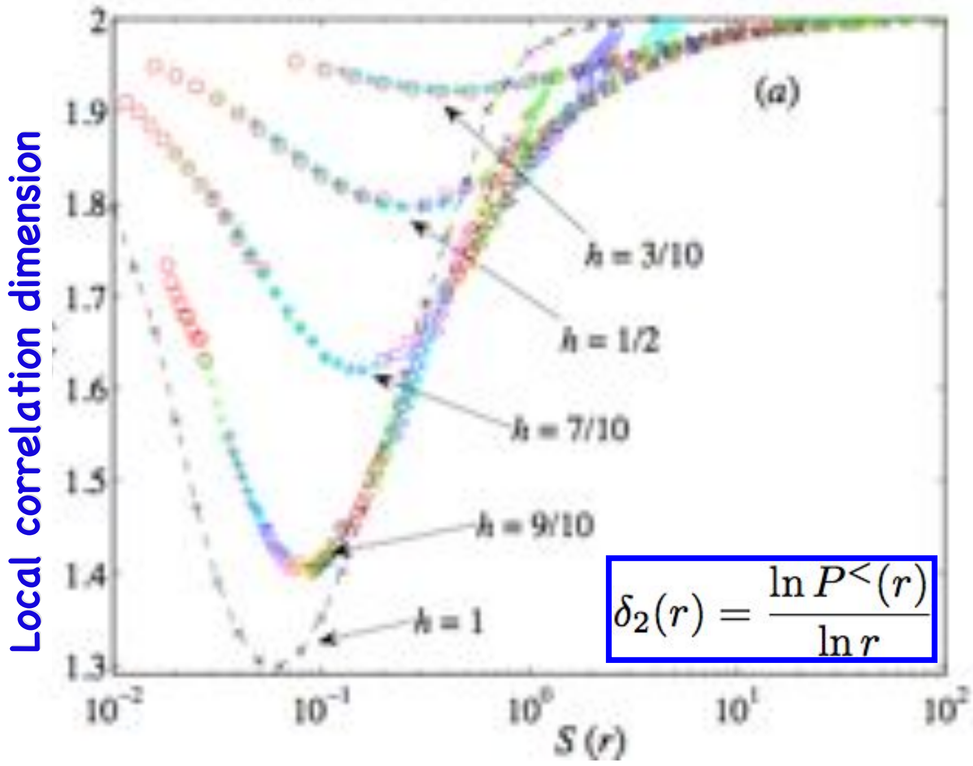
$$\ell \to \infty \Longrightarrow St(\ell) \to 0$$

Ballistic limit

$$\ell \to 0 \Longrightarrow St(\ell) \to \infty$$

Particle distribution is no more
Self-similar (fractal)
(Balkovsky, Falkovich, Fouxon 2001)

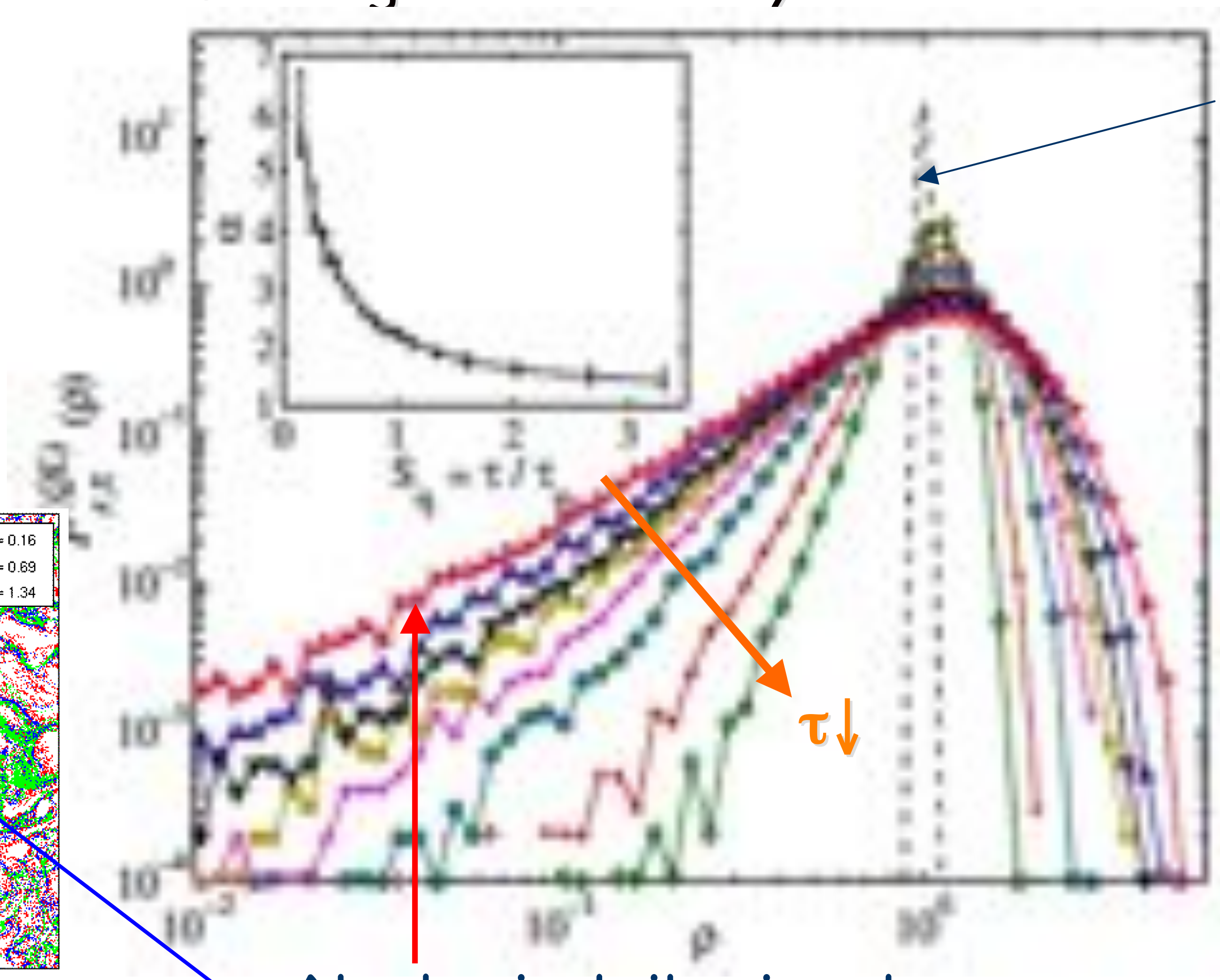
$$P_2(r) \sim r^{\delta_2(r)}$$



(Bec, MC & Hillenbrand 2007)

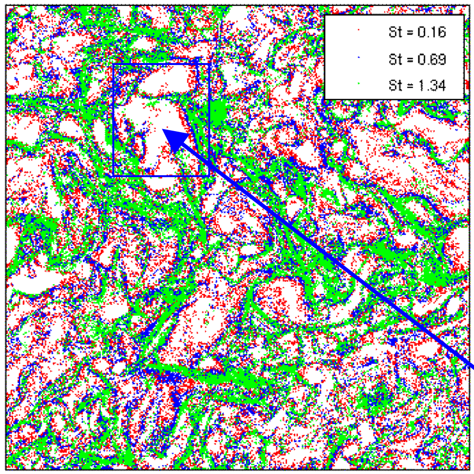
In turbulence?

Not enough scaling to study local dimensions
We can look at the coarse grained density



Poisson (τ=0)

$\tau\downarrow$



Algebraic tails signature of voids $p(\rho) \propto \rho^{\alpha(\tau,r)}$

What is the relevant time scale of inertial range clustering

For St->0 we have that

$$\boldsymbol{V} \approx \boldsymbol{u} - \tau D_t \boldsymbol{u} = u - \tau(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})$$

$$\boldsymbol{\nabla} \cdot V = -\tau \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}) = \boldsymbol{\tau} \nabla^2 p$$ Effective compressibility

We can estimate the phase-space contraction rate for
A particle blob of size r when the Stokes time is τ

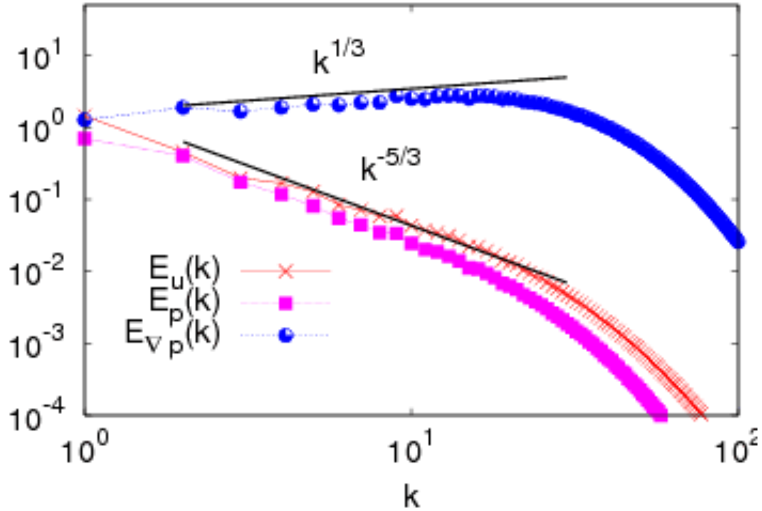
$$\boxed{\frac{1}{\mathcal{T}_{r,\tau}} = \frac{1}{r^3} \int_{[0:r]^3} d^3x \, \boldsymbol{\nabla} \cdot \boldsymbol{V} \sim -\frac{\tau \delta_r a}{r} \sim \frac{\tau \delta_r \nabla p}{r}}$$

It relates to pressure

Time scale of clustering

$$\frac{1}{\mathcal{T}_{r,\tau}} = \frac{1}{r^3}\int_{[0:r]^3} d^3x\,\nabla\cdot\boldsymbol{V} \sim -\frac{\tau\delta_r a}{r} \sim \frac{\tau\delta_r \nabla p}{r}$$

K41 $\delta_r \nabla p \approx \delta_r a \approx \frac{(\delta_r u)^2}{r} \approx \frac{\epsilon^{2/3}}{r^{1/3}} \Longrightarrow \begin{cases} E_p(k) \sim k^{-7/3} \\ E_{\nabla p}(k) \sim k^{-1/3} \end{cases} \Longrightarrow \mathcal{T}_{r,\tau} \approx \frac{r^{4/3}}{\tau\epsilon^{2/3}}$



Finite Re corrections on pressure spectra

experiments [Y. Tsuji and T. Ishihara (2003)]

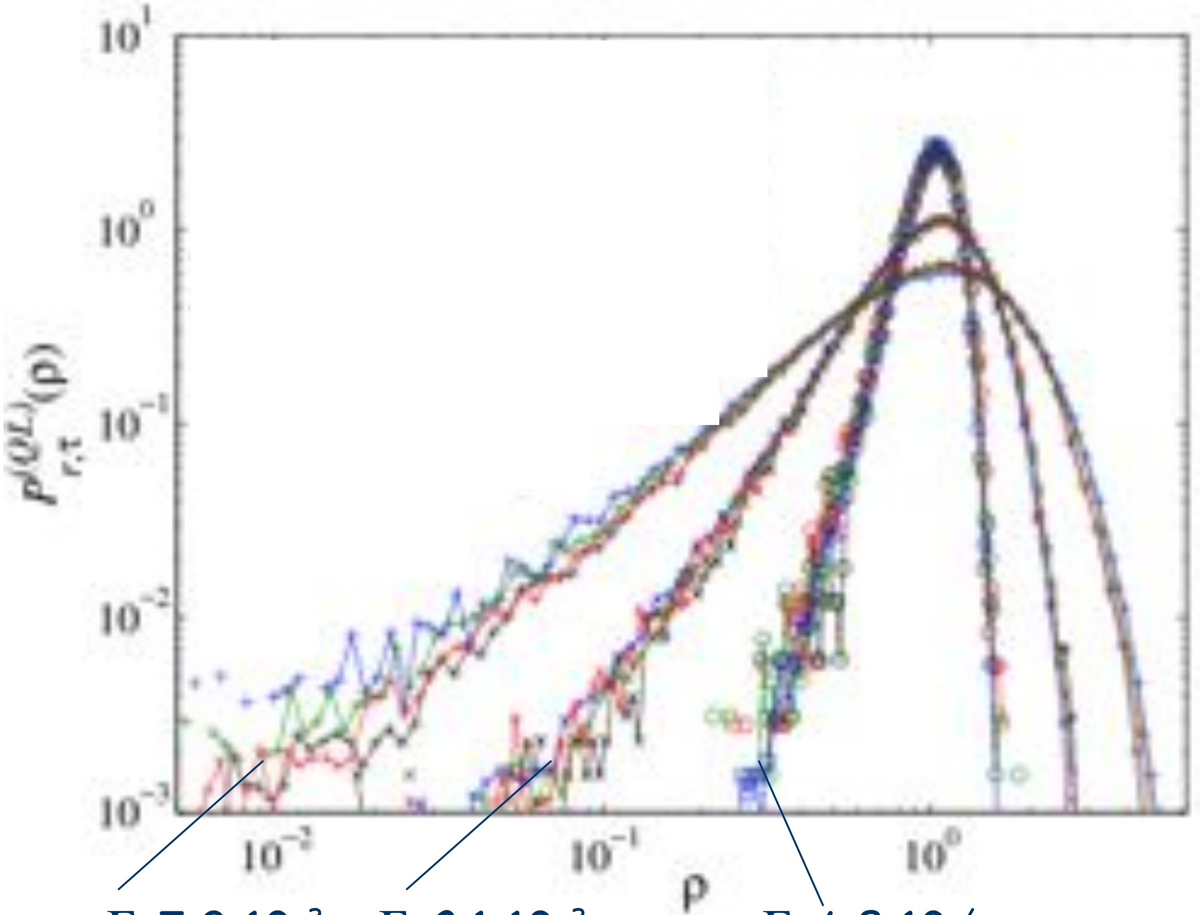
DNS [T. Gotoh and D. Fukayama (2001)]

Low Re- possible corrections due to sweeping

$\delta_r \nabla p \approx \delta_r a \approx \frac{U\delta_r u}{r} \approx \frac{U\epsilon^{1/3}}{r^{2/3}} \Longrightarrow \begin{cases} E_p(k) \sim k^{-5/3} \\ E_{\nabla p}(k) \sim k^{1/3} \end{cases} \Longrightarrow \mathcal{T}_{r,\tau} \approx \frac{r^{5/3}}{U\tau\epsilon^{1/3}}$

Nondimensional contraction rate

Adimensional contraction rate $\Gamma = \frac{\tau_\eta}{\mathcal{T}_{r,\tau}} \sim Re^{1/4}\, S_\eta \left(\frac{r}{\eta}\right)^{-5/3} \sim Re^{-1}\, S_\eta \left(\frac{r}{L}\right)^{-5/3}$



Γ=7.9 10^{-3} Γ=2.1 10^{-3} Γ=4.8 10^{-4}

Summary

- Clustering is a generic phenomenon in smooth flows: originates from dissipative dynamics (is present also in time uncorrelated flows)

- In time-correlated flows clustering and preferential concentration are linked phenomenon

- Tools from dissipative dynamical systems are appropriate for characterizing particle dynamics & clustering
  - Particles should be studied in their phase-space dynamics
  - Clustering is characterized by (multi)fractal distributions
  - Polydisperse suspensions can be treated similarly to monodisperse ones (properties depend on a length scale r^*)

- Time correlations are important in determining the properties very for small Stokes ($d_2-d \propto St^1$ or St^2, behavior of Lyapunov exponents)

- In the inertial range clustering is still present but is not scale invariant, in turbulence the coarse grained contraction rate seems to be the relevant time scale for describing clustering

Reading list

Stochastic flows

- M. Wilkinson & B. Mehlig, "Path coalescence transition and its application", PRE 68, 040101 (2003)
- J. Bec, "Fractal clustering of inertial particles in random flows", PoF 15, L81 (2003)
- J. Bec, "Multifractal concentrations of inertial particles in smooth random flows" JFM 528, 255 (2005)
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