Models of droplet collisions

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$$\frac{\partial n_i(r)}{\partial t} = \frac{1}{2}\sum_j \left\langle P_{i-j,j}(r)\right\rangle - \sum_j \left\langle P_{i,j}(r)\right\rangle$$

Smoluchowski equation

$$\frac{\partial n_i}{\partial t} = \frac{1}{2}\sum_j K_{j,i-j} n_j n_{i-j} - \sum_j K_{i,j} n_i n_j$$

$$\frac{\partial n(V)}{\partial t} = \frac{1}{2}\int dV' K(V', V-V') n(V') n(V-V') - n(V) \int dV' K(V, V') n(V')$$

$$\frac{\partial n(a)}{\partial t} = \int da' \left[\frac{K(a', a'') n(a'') n(a')}{2(a''/a)^2} - K(a', a) n(a') n(a) \right]$$

$$a'' = (a^3 - a'^3)^{1/3}$$

Collision kernel

$$\frac{\partial n_i}{\partial t} = \frac{1}{2}\sum_j K_{j,i-j} n_j n_{i-j} - \sum_j K_{i,j} n_i n_j$$

$$K(a, a', g, \rho_w, \rho_a, \eta_w, \eta_a, T, \alpha)$$

$$[K] = [1/nt] = cm^3 \sec^{-1}$$

$$K = \frac{\rho_w g a^2}{\eta_a} f\left(\frac{a}{a'}, \frac{\rho_a}{\rho_w}, \frac{\eta_a}{\eta_w}, \frac{\rho_w g a^2}{\alpha}, \frac{\lambda}{a}\right)$$

Brownian Coagulation

$$n_2(r) = n_2(\infty)\left(1 + \frac{a_1 + a_2}{r}\right)$$

$$J = 4\pi(a_1 + a_2)(D_1 + D_2)n_2$$

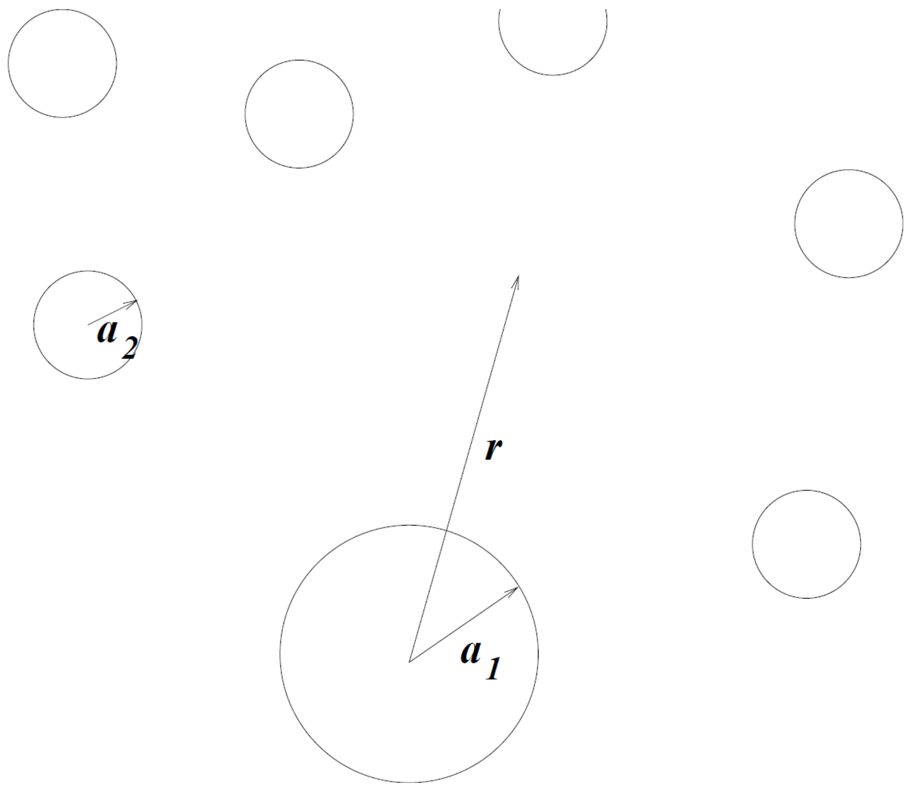
$$\frac{\partial n_2}{\partial t} = \frac{\partial n_1}{\partial t} = -4\pi(a_1 + a_2)(D_1 + D_2)n_1 n_2 = -K_{12} n_1 n_2$$

$$D(a_i) = \frac{kT}{6\pi\eta_a a_i}$$

$$K_{12} = \frac{2kT}{3\eta_a}\frac{(a_1 + a_2)^2}{a_1 a_2}$$

$$1/2 < a_1/a_2 < 2$$

$$K_{12} \approx \frac{8kT}{3\eta_a}$$



$$\frac{\partial n_i}{\partial t} = \frac{K}{2}\sum_j n_j n_{i-j} - K\sum_j n_i n_j$$

$$\frac{d}{dt}\sum_i n_i(t) = \frac{dN}{dt} = -KN^2/2$$

$$N(t) = \frac{N(0)}{1 + KN(0)t/2}$$

$$\frac{\partial n_1}{\partial t} = -Kn_1 N(t) \;\Rightarrow\; n_1(t) = \frac{n_1(0)}{[1 + KN(0)t/2]^2}$$

$$n_i(0) = N(0)\delta_{i0}\,, \quad n_i(t) = \frac{N(0)(KN_0 t)^{i-1}}{[1 + KN(0)t/2]^{i+1}}$$

$$\frac{\partial n(a)}{\partial t} = \int da' \left[\frac{K(a',a'')n(a'')n(a')}{2(a''/a)^2} - K(a',a)n(a')n(a) \right]$$

$$n(a,t) = t^{-q} f(at^{-p})$$

$$\int a^3 n(a)da = t^{4p-q} \int x^3 f(x)dx \Rightarrow q = 4p$$

$$\partial n / \partial t \propto t^{-q-1} f \ldots, \int Knnda \propto t^{-2q+p(\alpha+1)} \Rightarrow q - 1 = p(\alpha+1)$$

$$p = \frac{1}{3-\alpha}, q = \frac{4}{3-\alpha}.$$

$$\alpha > 3 \Rightarrow n(a,t) = (t_0 - t)^{-q} f[a(t_0 - t)^{-p}]$$

$$K(\lambda a, \lambda a') = \lambda^\alpha K(a,a')$$

$\alpha > 3$ Accelerating propagation towards large sizes

$\alpha < 3$ Decelerating propagation towards large sizes

$\alpha = 3$ Uniform propagation towards large sizes collision kernel

$$K(a, a') \simeq \pi (a + a')^2 \Delta v$$

$$K = K_{\mathrm{g}} + K_t$$

gravity collision kernel

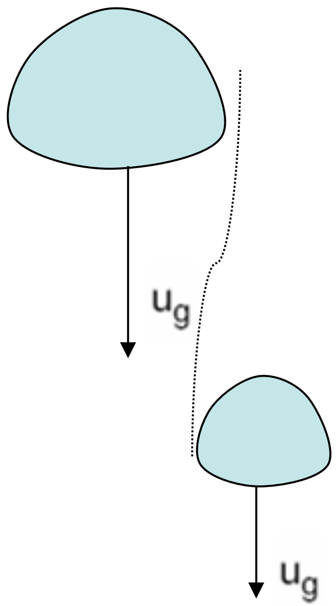
Collision rate = target area $\pi(a + a')^2$ times velocity difference

$$K_{\mathrm{g}}(a, a') = \pi (a + a')^2 E(a, a') |u_{\mathrm{g}}(a) - u_{\mathrm{g}}(a')|$$

Collisions

Settling velocity is obtained from the force balance:

gravity force $(4\pi a^3/3)\rho_0 g$ equal to viscous friction $6\pi\nu\rho u_g a$



$$u_g = g\tau$$

$$\tau = (2/9)(\rho_0/\rho)(a^2/\nu)$$

Fall velocity of small droplets

$$6\pi R\eta_a u = mg$$

$$R = 0.01\, mm = 10\, \mu m$$

$$u = \frac{2\rho_w g R^2}{9\eta_a} \simeq 1.21\,\mathrm{cm/s}$$

$$\mathrm{Re} \simeq 0.008 \qquad\qquad \mathrm{Re} \propto vR \propto R^3$$

$$\mathrm{Re} \simeq 1 \text{ already for } R = 0.05\,\mathrm{mm}$$

$$\eta_a = 1.8 \cdot 10^{-4}\,\mathrm{g/s\cdot cm},\ \eta_w = 0.01\ \mathrm{g/s\cdot cm}$$

$$\rho_w = 1\,\mathrm{g/cm^3} \qquad \rho = 1.2\cdot 10^{-3}\,\mathrm{g/cm^3}$$

Sphericity

viscous stress $\eta_w u/R$

surface tension stress α/R

$$\eta_w u/\alpha \simeq 0.00017 \text{ for } \alpha = 70\,g/s^2$$

$$R = 0.01\,mm = 10\,\mu m$$

Internal circulation

$$\eta_a/\eta_w \simeq 0.018$$

$$F = 2\pi u \eta_a R \frac{2\eta_a + 3\eta_w}{\eta_a + \eta_w}$$

$$u = \frac{2\rho R^2 g}{3\eta_a}\left(\frac{3\eta_a + 3\eta_w}{2\eta_a + 3\eta_w}\right)$$

$$\simeq \frac{2\rho R^2 g}{9\eta_a}\left(1 + \frac{1}{3}\frac{\eta_a}{\eta_w}\right) \simeq 1.22\,\frac{\text{cm}}{\text{s}}$$

$$u_g = \frac{2(\rho_w - \rho_a)ga^2}{9\eta_a}\left[1 + O\left(\frac{\lambda}{a}\right) - O\left(\frac{\eta_w \rho_w g a^2}{\eta_a \alpha}\right) - O\left(\frac{\rho_w \rho_a g a^3}{\eta_a^2}\right)\right]$$

$Re \lesssim 1$

$$u_g = g\tau_p / f(Re)$$

$$\tau_p \equiv 2\rho_w a^2 / 9\rho\nu$$

$$f(Re_p) = 1 + 0.15 Re_p^{0.687}$$

$Re \gg 1$

$$F = C\rho_a \pi a^2 u_g^2 = \frac{4\pi a^3}{3}\rho_w g$$

$$u_g = \sqrt{\frac{4\rho_w g a}{3\rho_a C}}$$

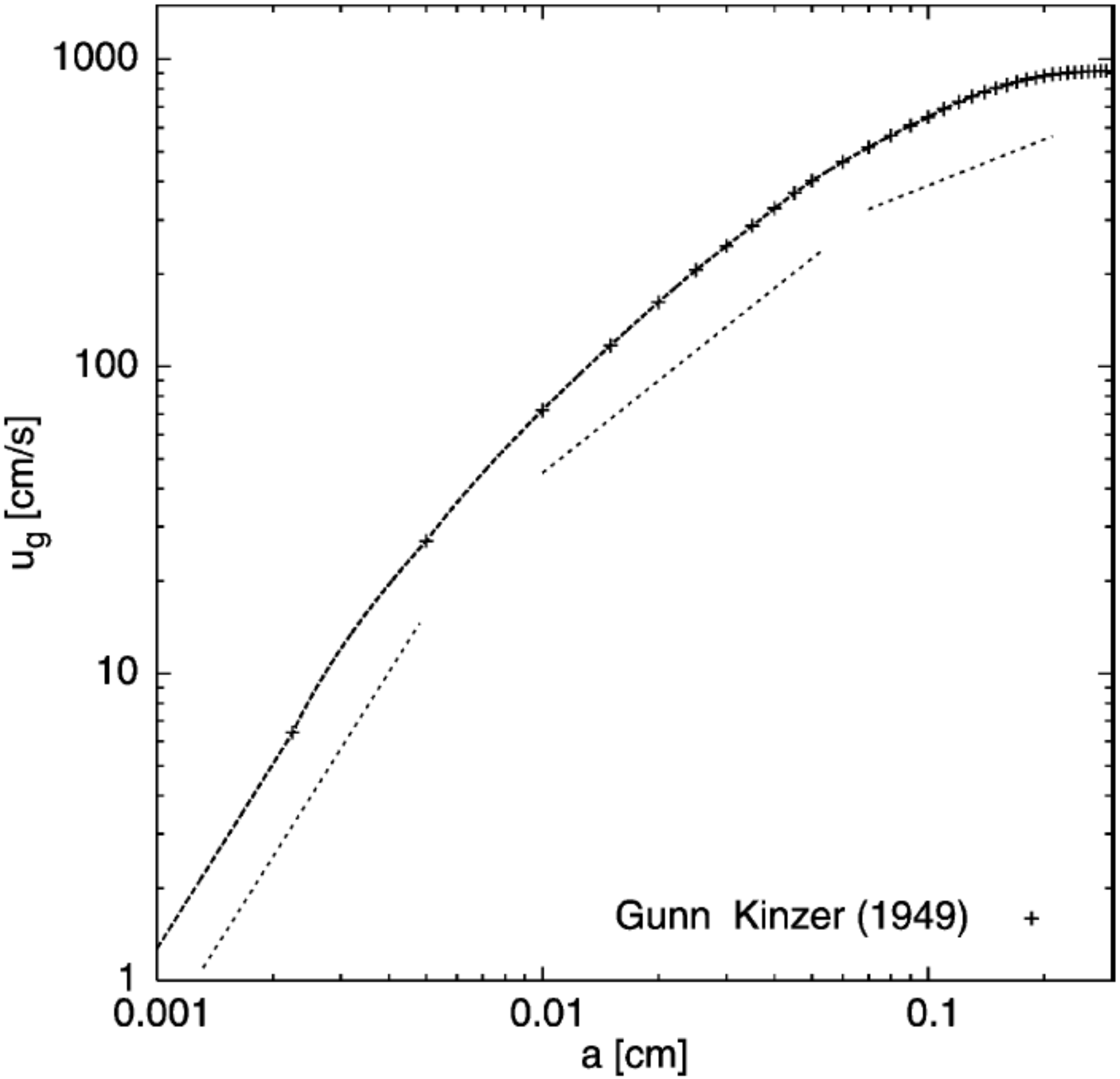
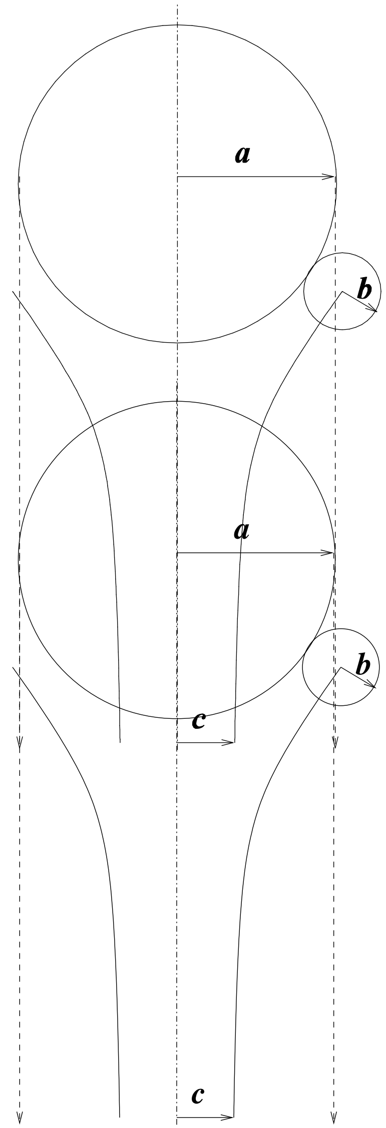


FIG. 1. Terminal fall velocity $\mathbf{u}_g$ as a function of cloud droplet radius a.

Collision efficiency is the ratio of the actual collision cross-section to the geometrical cross-section

a, b are droplet radii

c is the critical offset for a grazing trajectory of the small droplet



$$E(a, b) = \frac{c^2}{a^2 + b^2}$$

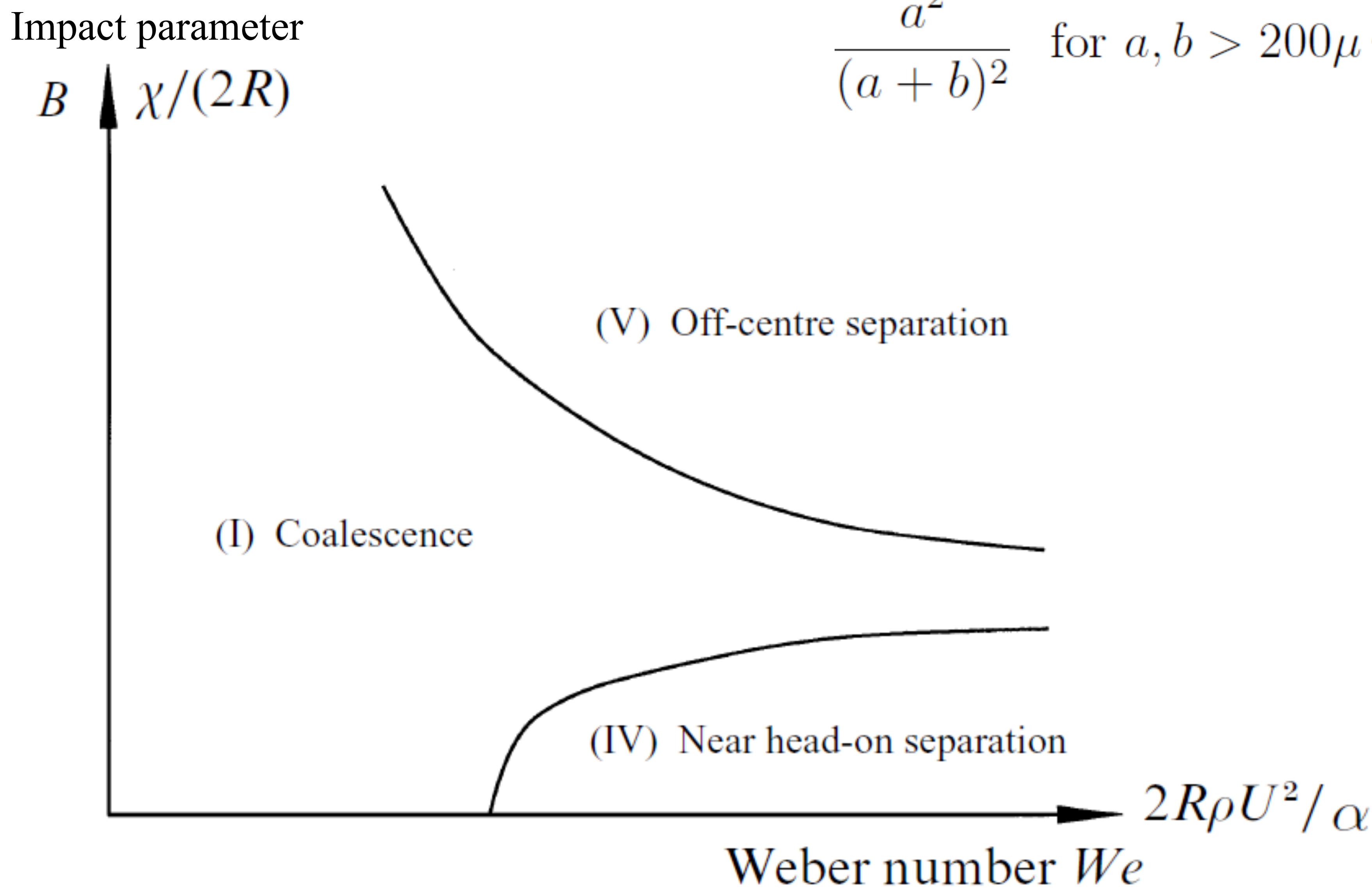
$$E(b/a, Re) \Rightarrow c = af(b/a, Re)$$

$$\lim_{a/b \to 0, Re \to 0} E(b/a, Re) = \frac{b^2}{2(a^2 + b^2)} \approx \frac{b^2}{2a^2}$$

$$\lim_{a/b \to 0, Re \to 0} K_g(a, b) = \pi \frac{b^2}{2a^2} u_g(a) = \pi \frac{2gb^2}{9\nu} \frac{\rho_w}{\rho_a}$$

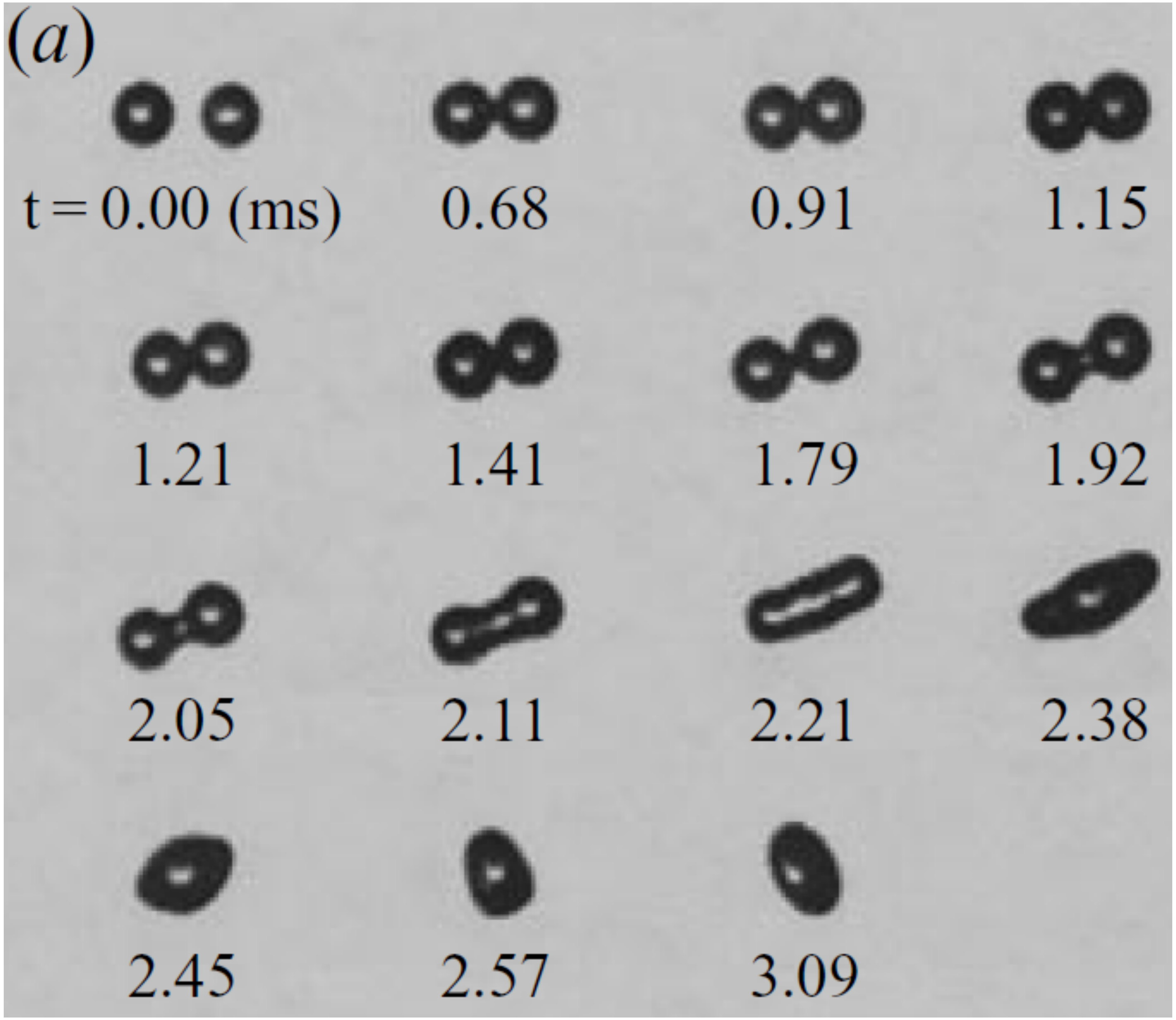
Coalescence efficiency

$$\frac{a^2}{(a+b)^2} \quad \text{for } a, b > 200\mu m$$

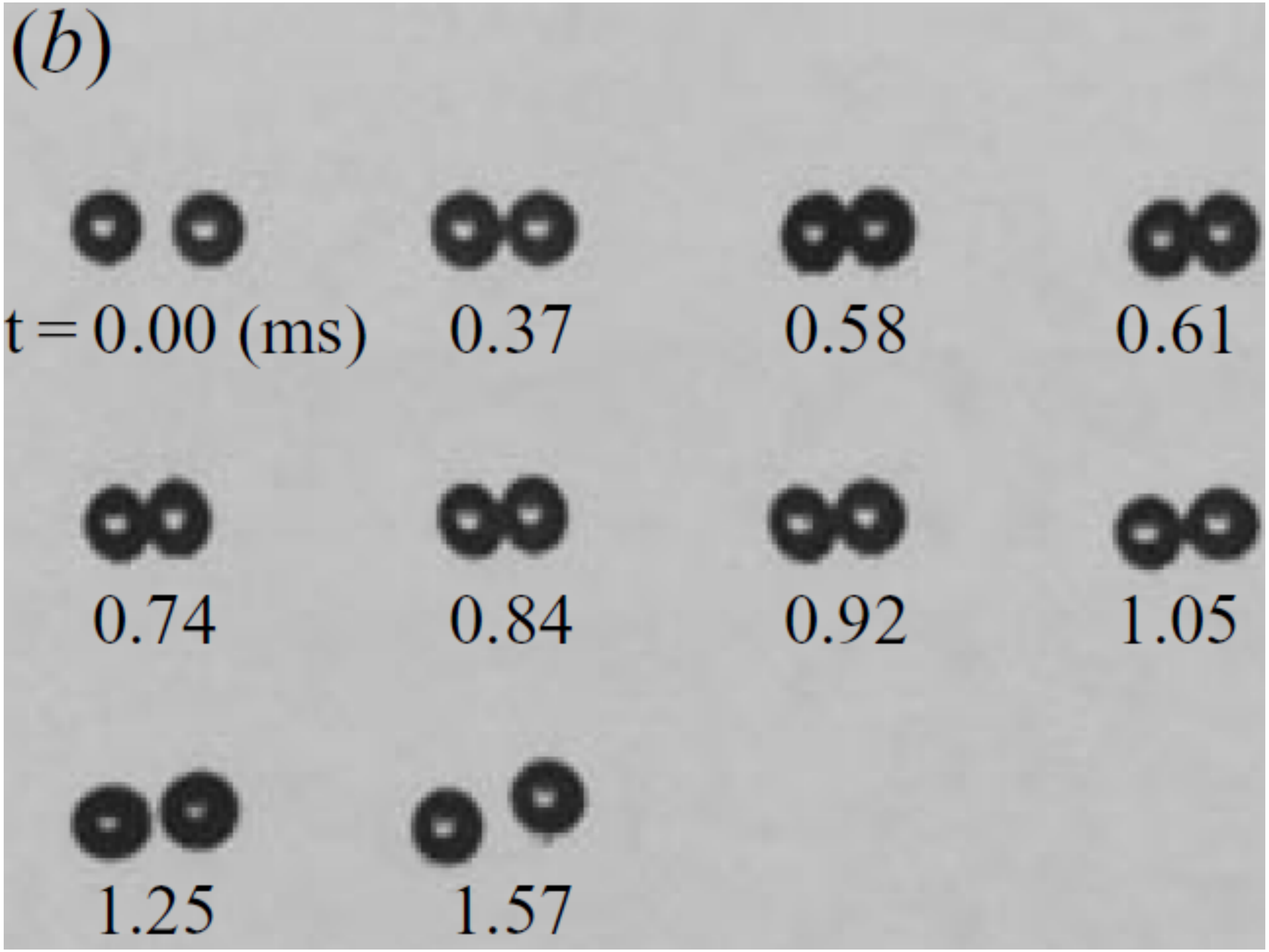


Schematic of various collision regimes of water droplets in 1 atm. air.

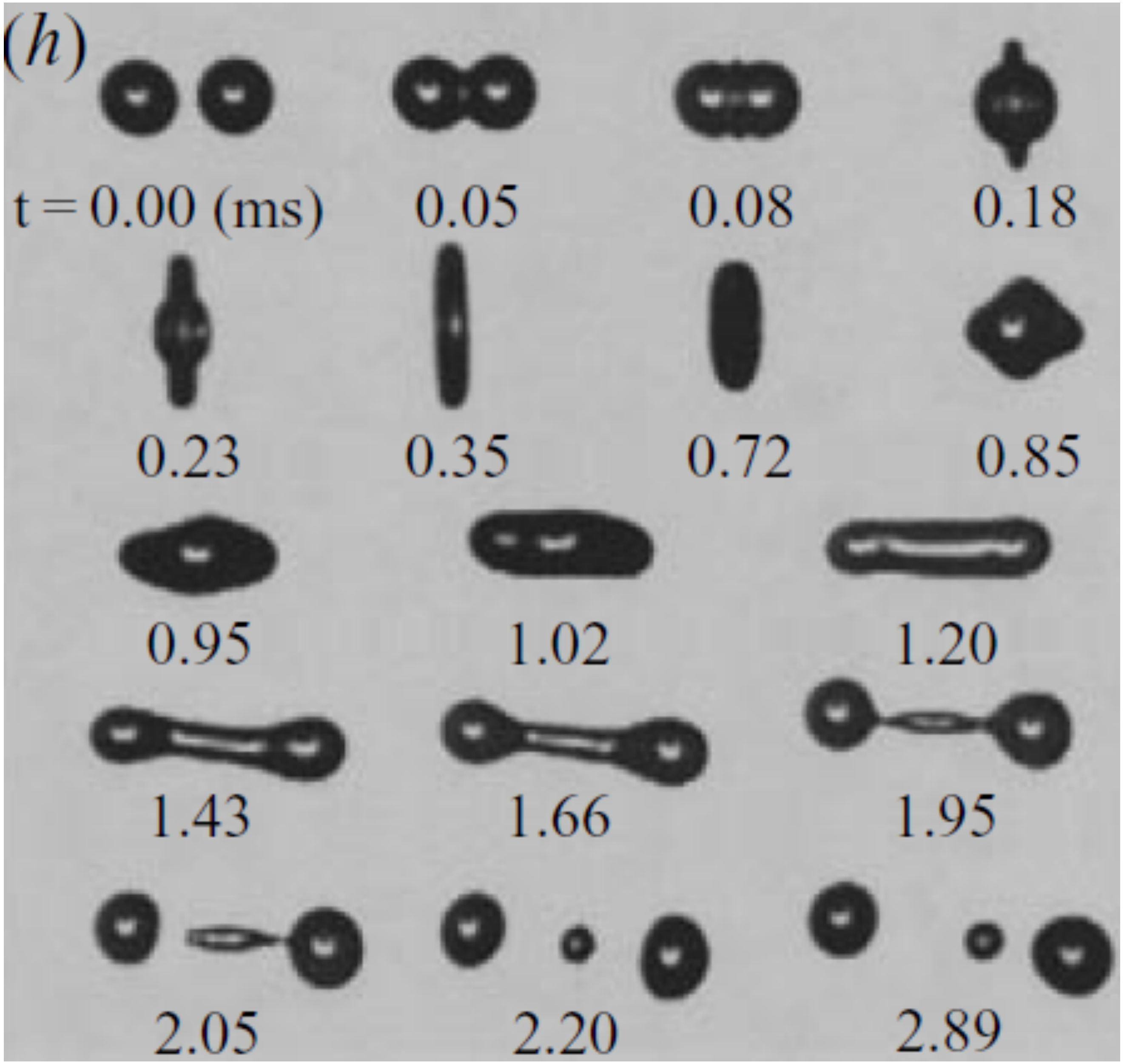
Weber number slightly below the threshold for coalescence

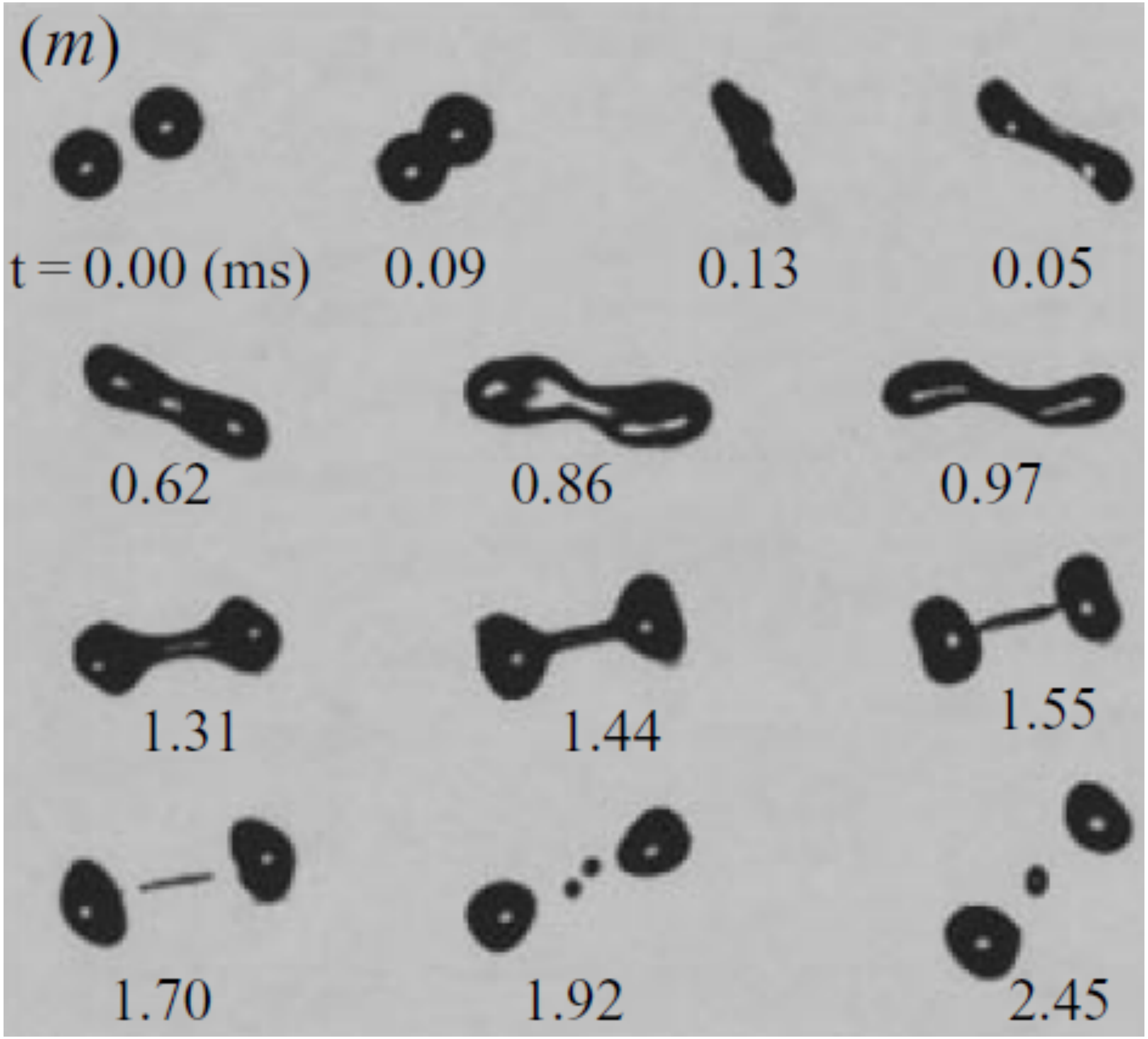


Weber number slightly above the threshold for coalescence



Break-up


(h)
t = 0.00 (ms)
0.05
0.08
0.18
0.23
0.35
0.72
0.85
0.95
1.02
1.20
1.43
1.66
1.95
2.05
2.20
2.89

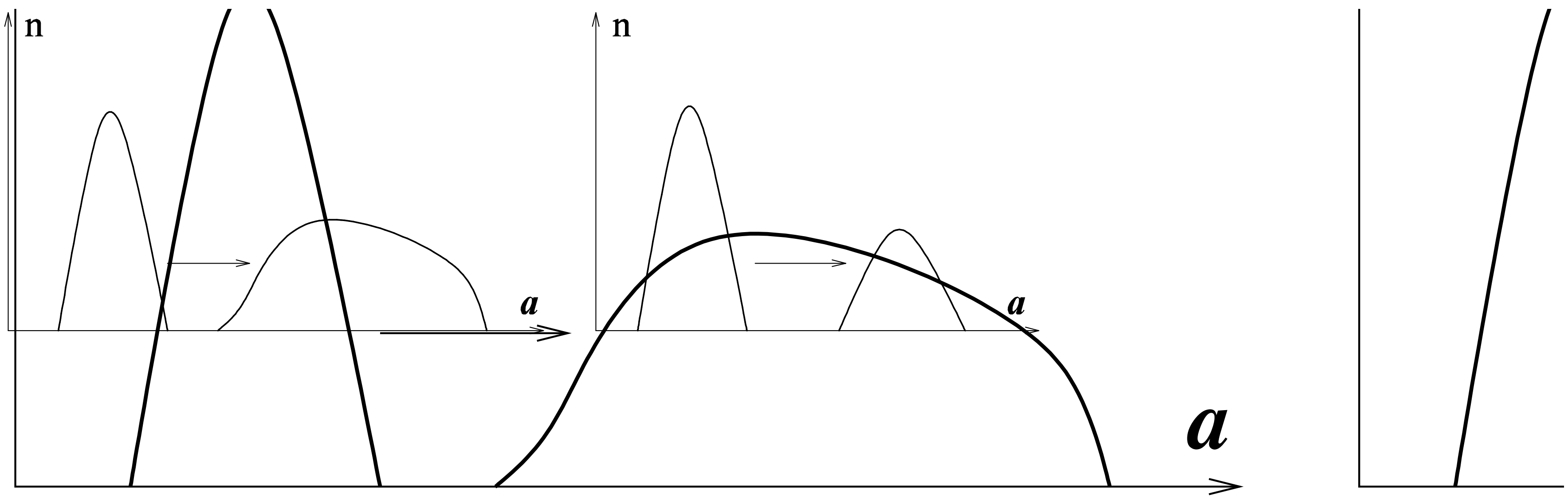

(m)
t = 0.00 (ms)
0.09
0.13
0.05
0.62
0.86
0.97
1.31
1.44
1.55
1.70
1.92
2.45

Lecture 2

Smoluchowski equation

$$\frac{\partial n(a)}{\partial t} = \int da' \left[\frac{K(a',a'')n(a'')n(a')}{2(a''/a)^2} - K(a',a)n(a')n(a) \right]$$

$$a'' = (a^3 - a'^3)^{1/3}$$



?

$$n(a,t) = t^{-q} f(at^{-p})$$

$$\int a^3 n(a)da = t^{4p-q}\int x^3 f(x)dx \Rightarrow q = 4p$$

$$\partial n/\partial t \propto t^{-q-1} f \ldots, \int Knnda \propto t^{-2q+p(\alpha+1)} \Rightarrow q-1 = p(\alpha+1)$$

$$p = \frac{1}{3-\alpha}, q = \frac{4}{3-\alpha}.$$

$$\alpha > 3 \Rightarrow n(a,t) = (t_0 - t)^{-q} f[a(t_0 - t)^{-p}]$$

$$K(\lambda a, \lambda a') = \lambda^{\alpha} K(a,a')$$

$\alpha > 3$ Accelerating propagation towards large sizes

$\alpha < 3$ Decelerating propagation towards large sizes

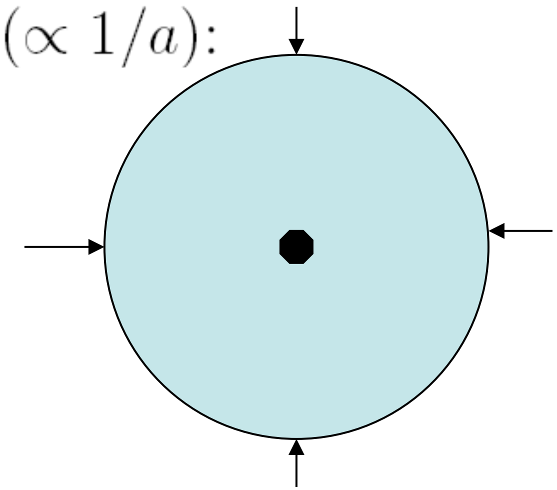
$\alpha = 3$ Uniform propagation towards large sizes

$$K_B(a_1,a_2) \propto (a_1 + a_2)^2 / a_1 a_2 \Rightarrow \alpha = 0$$

$$K_{\mathrm{g}}(a, a') = \pi(a+a')^2 E(a, a')|u_{\mathrm{g}}(a) - u_{\mathrm{g}}(a')|$$

Volume growth rate ($\propto a^3/t$) is due to the flux

area ($\propto a^2$) times vapour concentration gradient ($\propto 1/a$):

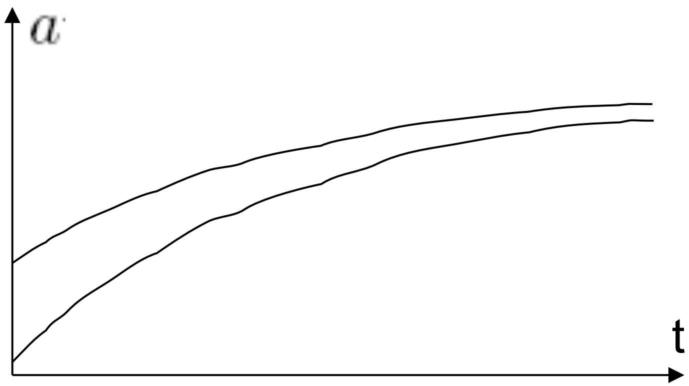


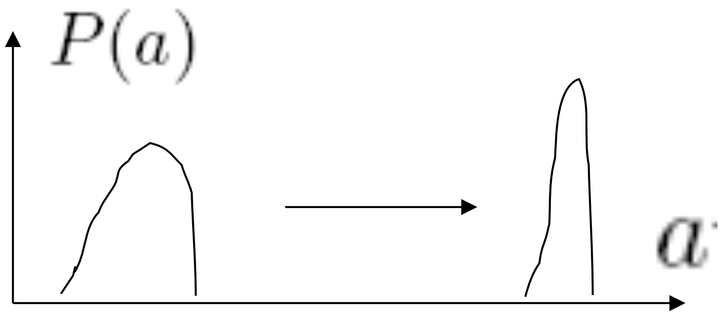
$$a^3/t \propto a^2/a \qquad \Rightarrow \qquad a^2 \propto t$$

$$\frac{4\pi}{3}\rho_0 \frac{da^3}{dt} = \text{flux} = 4\pi\kappa(M - M_s)a$$

$$\frac{da^2}{dt} = \frac{2\kappa(M - M_s)}{\rho_0}$$

Realistic model
in Grabowski lectures





Usually condensational growth dominates over
Brownian coalescence

Condensation slows down while gravitational collisions initially accelerate with time (with the growth of droplet size) so there must be a crossover size that can be estimated from the explicit relation

$$K(a_c)n_0 \simeq \kappa s M/\rho_0 a_c^2$$

Bottleneck at the crossover size which determines the typical time of growth from 1 μm to 100 μm

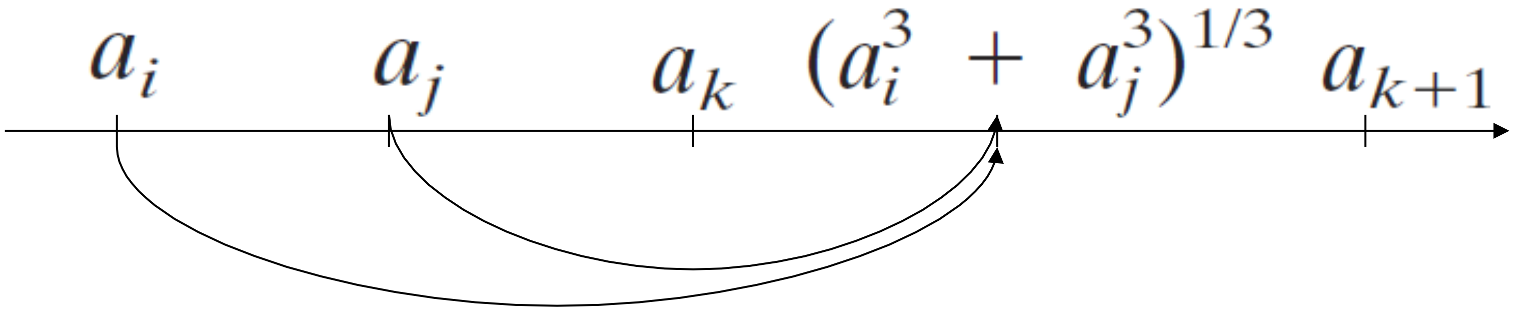
Discrete conservative scheme of calculating collisions

$$\delta n_i = \delta n_j = -dN = -\delta n_k - \delta n_{k+1},$$

$$a_k^3 \delta n_k + a_{k+1}^3 \delta n_{k+1} = (a_i^3 + a_j^3) dN,$$

$$\delta n_{k+1} = dN(a_i^3 + a_j^3 - a_k^3)/(a_{k+1}^3 - a_k^3)$$

$$\delta n_k = dN(a_{k+1}^3 - a_i^3 + a_j^3)/(a_{k+1}^3 - a_k^3)$$



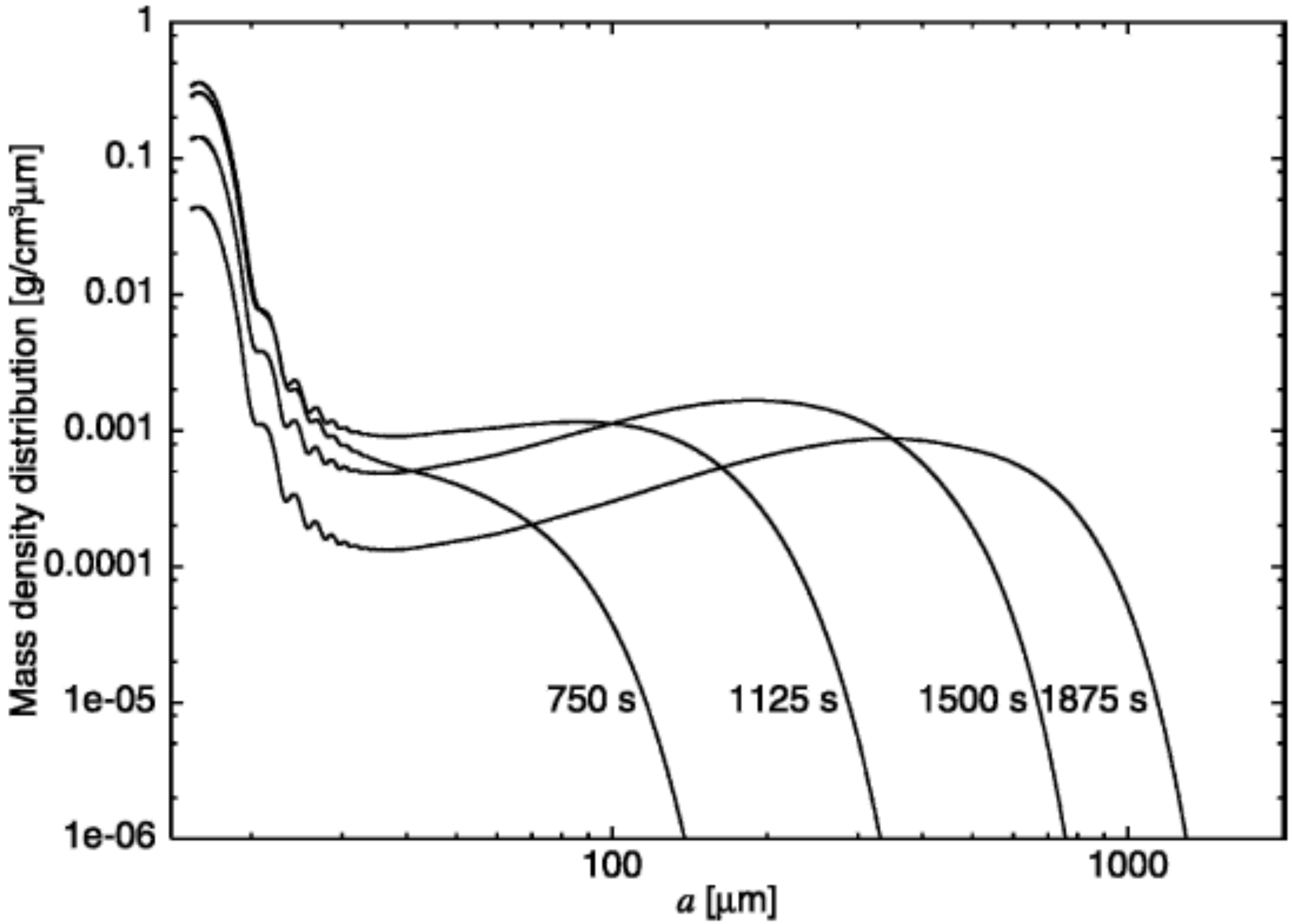


FIG. 4. Mass density of water shown at different moments in time and as a function of droplet radii a. Rain initiation time is $t_* \simeq 1500$ s. Notice how with the evolution of time the largest amount of droplets moves from small radii to larger ones.

$$K_{\mathrm{g}}(a, a') = \pi (a + a')^2 E(a, a') |u_{\mathrm{g}}(a) - u_{\mathrm{g}}(a')|$$

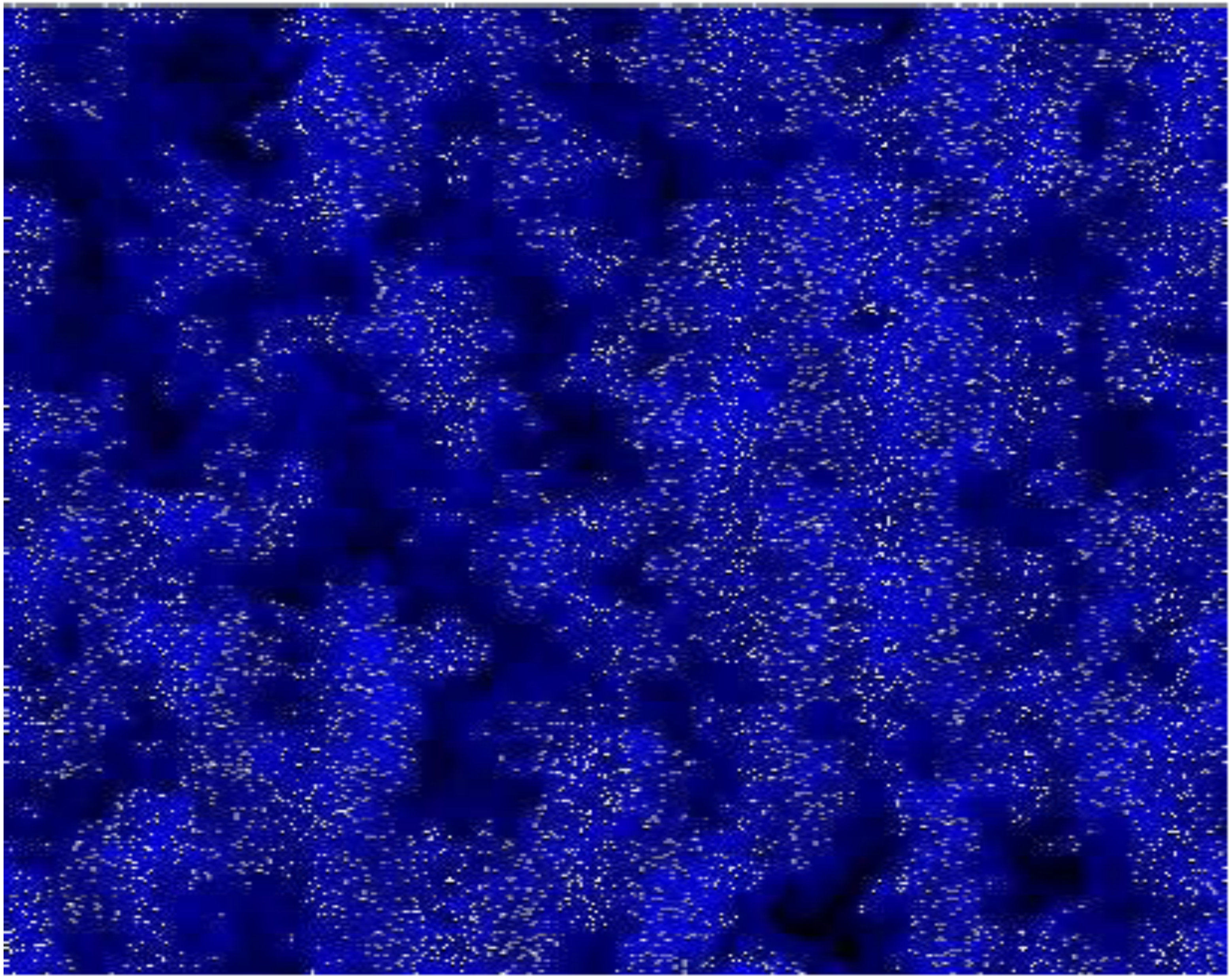
Turbulence characteristics

$$\eta = \left(\nu^3/\epsilon\right)^{1/4}, \qquad v_k = \left(\nu\epsilon\right)^{1/4}, \qquad \tau_k = \left(\nu/\epsilon\right)^{1/2}$$

$$K_t(a, a', \rho_w, \rho_a, \eta_w, \eta_a, \epsilon, L) = \frac{\rho_w g a^2}{\eta_a} f(a/\eta, St, Sv, Re)$$

$$St = \frac{\tau_p}{\tau_k}, \quad Sv = \frac{v_p}{v_k}$$

See Cencini and Grabowski lectures



Spatial distribution of droplets (white points) and supersaturation field (blue).

ϵ ($\mathrm{cm^2/s^3}$)	τ_k (s)	η (cm)	υ_k (cm/s)
10	0.1304	0.1488	1.142
100	0.0412	0.0837	2.031
400	0.0206	0.0592	2.872

Ayala, Rosa, Wang, Grabowski

Table 2. Basic properties of cloud droplets.

a (μm)	τ_p (s)	v_p (cm/s)	Re_{p0}	$f(Re_{p0})$
10	0.0013	1.272	0.015	1.008
20	0.0052	4.959	0.116	1.034
30	0.0118	10.717	0.378	1.077
40	0.0209	18.089	0.851	1.134
50	0.0327	26.624	1.566	1.204
60	0.0471	35.944	2.537	1.284

Table 3. Characteristic scales of cloud droplets.

a	$\epsilon \quad (cm^2/s^3)$								
	10			100			400		
(μm)	St	Sv	a/η	St	Sv	a/η	St	Sv	a/η
10	0.010	1.113	0.007	0.032	0.626	0.011	0.063	0.442	0.017
20	0.040	4.343	0.013	0.127	2.442	0.024	0.253	1.727	0.034
30	0.090	9.385	0.020	0.285	5.278	0.036	0.570	3.732	0.051
40	0.160	15.841	0.027	0.507	8.908	0.047	1.014	6.299	0.067
50	0.250	23.316	0.033	0.792	13.111	0.059	1.585	9.271	0.084
60	0.361	31.478	0.040	1.141	17.701	0.071	2.282	12.516	0.101

$$\varepsilon \approx u^3 / L$$

Ratio of turbulent collision kernel to gravity kernel

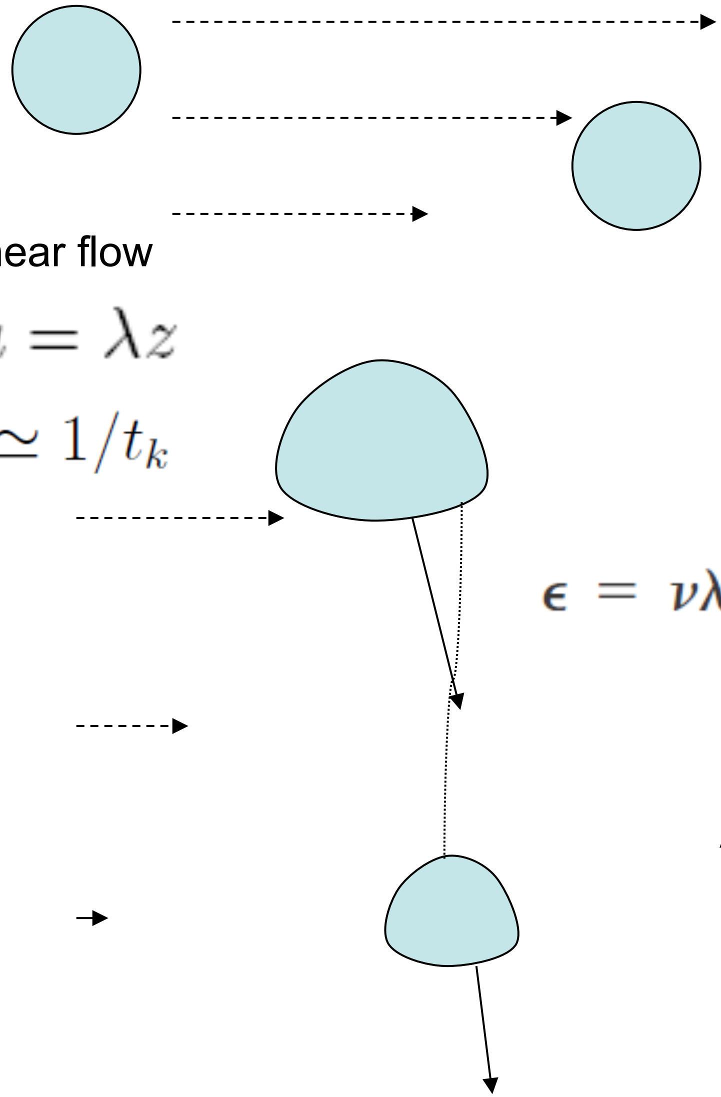
		ϵ 100 cm^2/s^3			400 cm^2/s^3		
		R_λ			R_λ		
a_1 (μm)	a_2 (μm	23.4	43.0	72.4	23.4	43.0	72.4
10	20	1.035 (0.011)	1.031 (0.014)	1.061 (0.019)	1.070 (0.011)	1.136 (0.013)	1.145 (0.016)
10	30	1.019 (0.004)	1.032 (0.004)	1.029 (0.013)	1.064 (0.002)	1.092 (0.003)	1.113 (0.008)
10	40	1.016 (0.005)	1.020 (0.011)	1.006 (0.012)	1.053 (0.005)	1.083 (0.011)	1.090 (0.013)
10	50	1.020 (0.005)	1.018 (0.010)	1.033 (0.010)	1.037 (0.006)	1.046 (0.010)	1.070 (0.011)
10	60	1.007 (0.003)	1.018 (0.009)	1.020 (0.011)	1.020 (0.002)	1.028 (0.007)	1.053 (0.012)
20	30	1.058 (0.011)	1.070 (0.011)	1.076 (0.015)	1.172 (0.012)	1.263 (0.017)	1.314 (0.015)
20	40	1.038 (0.008)	1.045 (0.011)	1.040 (0.011)	1.061 (0.006)	1.127 (0.015)	1.123 (0.018)
20	50	1.013 (0.005)	1.026 (0.010)	1.012 (0.013)	1.029 (0.005)	1.047 (0.011)	1.068 (0.014)
20	60	1.007 (0.003)	1.005 (0.009)	1.006 (0.012)	1.013 (0.003)	1.030 (0.008)	1.042 (0.013)

$$K(a,a') = \pi(a+a')^2 \Delta v P(a+a')$$

$$St \ll 1, \quad Sv \ll 1$$

$$P(2a) = 1$$

$$\Delta v \sim \lambda a$$



Shear flow

$$u = \lambda z$$

$$\lambda \simeq 1/t_k$$

$$K_s(a,a') = \lambda\pi(a+a')^3$$

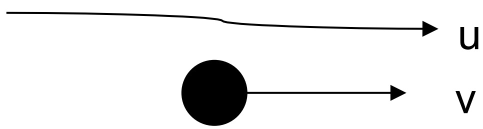
$$\epsilon = \nu\lambda^2 \simeq 10^2 \div 10^3 \ \mathrm{cm^2\,s^{-3}}$$

$$a \lesssim 5\ \mu\mathrm{m}$$

Additional collision kernel (Saffman&Turner)

Change in collision efficiency (Khain&Pinsky)

Inertial particles



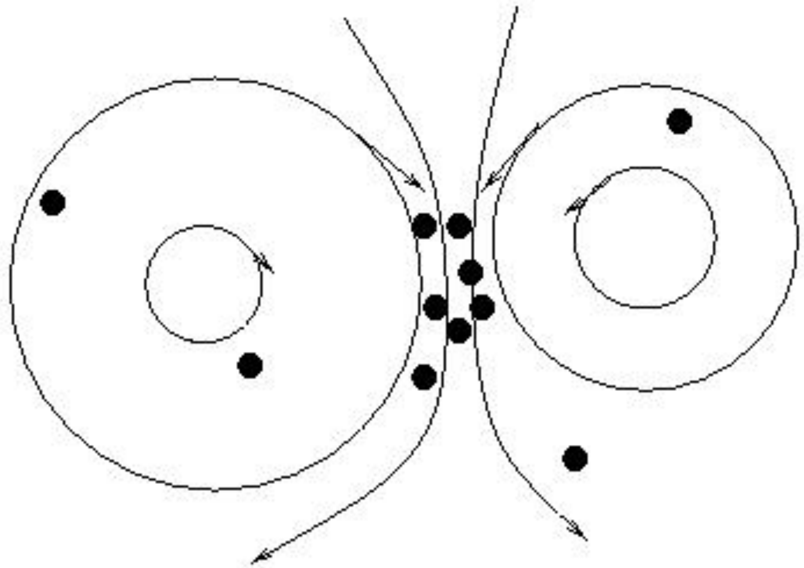
$$\frac{d\mathbf{v}}{dt} - \beta\frac{d\mathbf{u}}{dt} = \frac{\mathbf{u}-\mathbf{v}}{\tau} + \mathbf{g} \qquad \tau \; = \; (2/9)(\rho_0/\rho)(a^2/\nu)$$

$$\mathbf{v} = \mathbf{u} + (\beta - 1)\tau[\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}] + \mathbf{g}\tau \qquad \beta = 3\rho/(\rho + 2\rho_0)$$

$$\nabla\cdot\mathbf{v} \; = \; (\beta-1)\tau\nabla[(\mathbf{u}\cdot\nabla)\mathbf{u}] \qquad \nabla[(\mathbf{u}\cdot\nabla)\mathbf{u}] \; = \; S^2 - \Omega^2$$

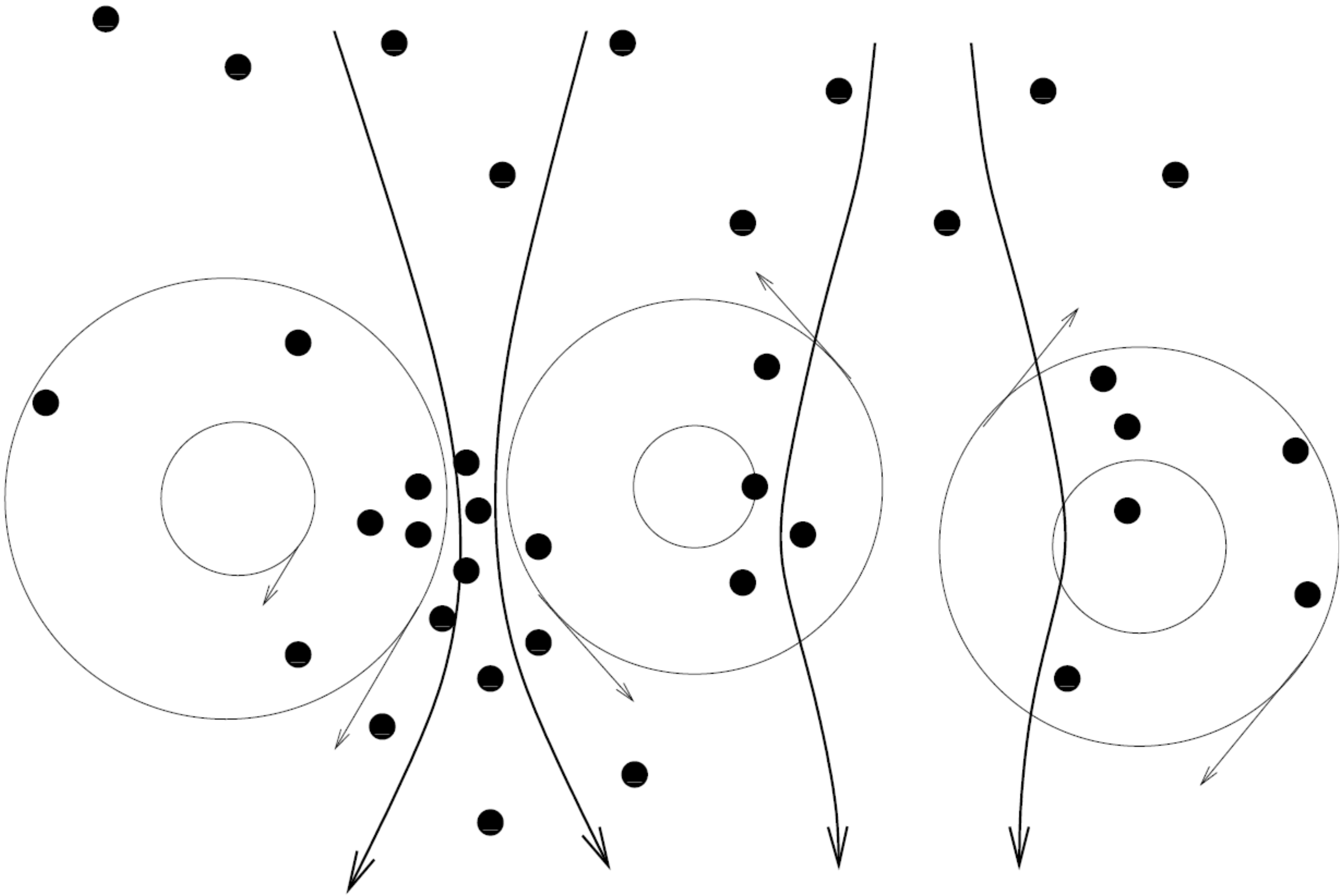
$$St = \tau|\nabla u| \simeq \tau/\tau_K$$

Maxey & Wang

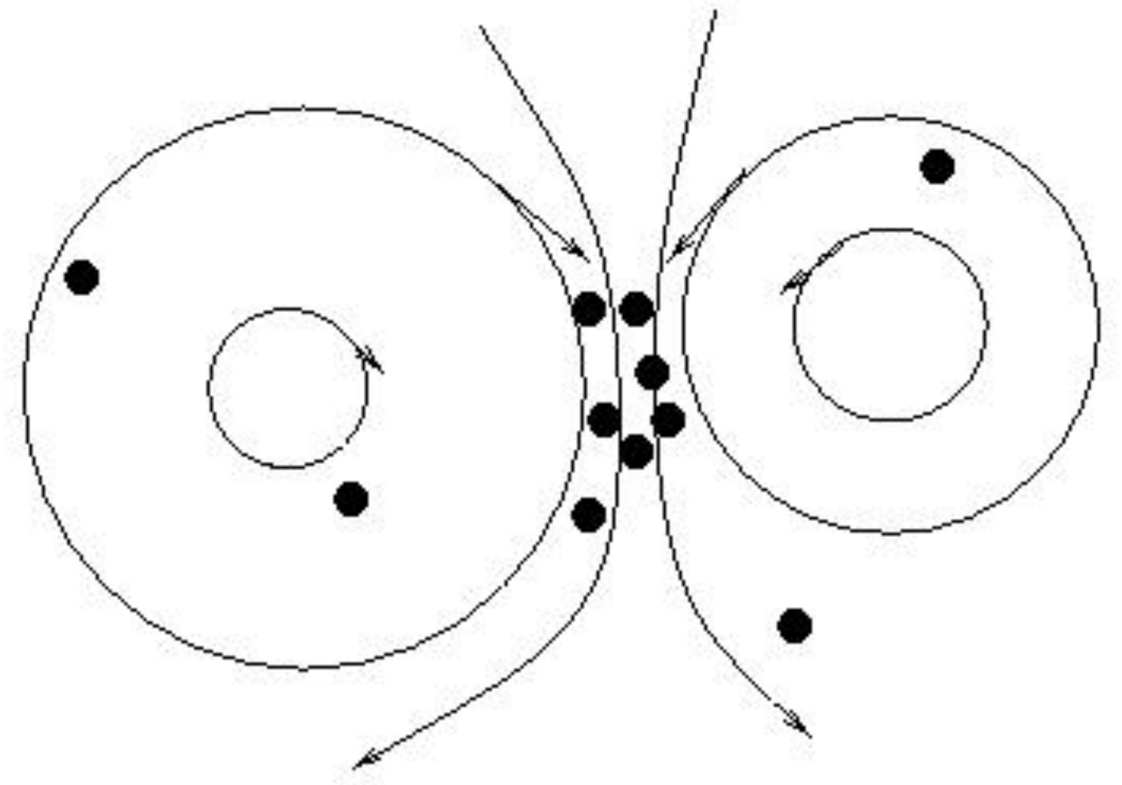


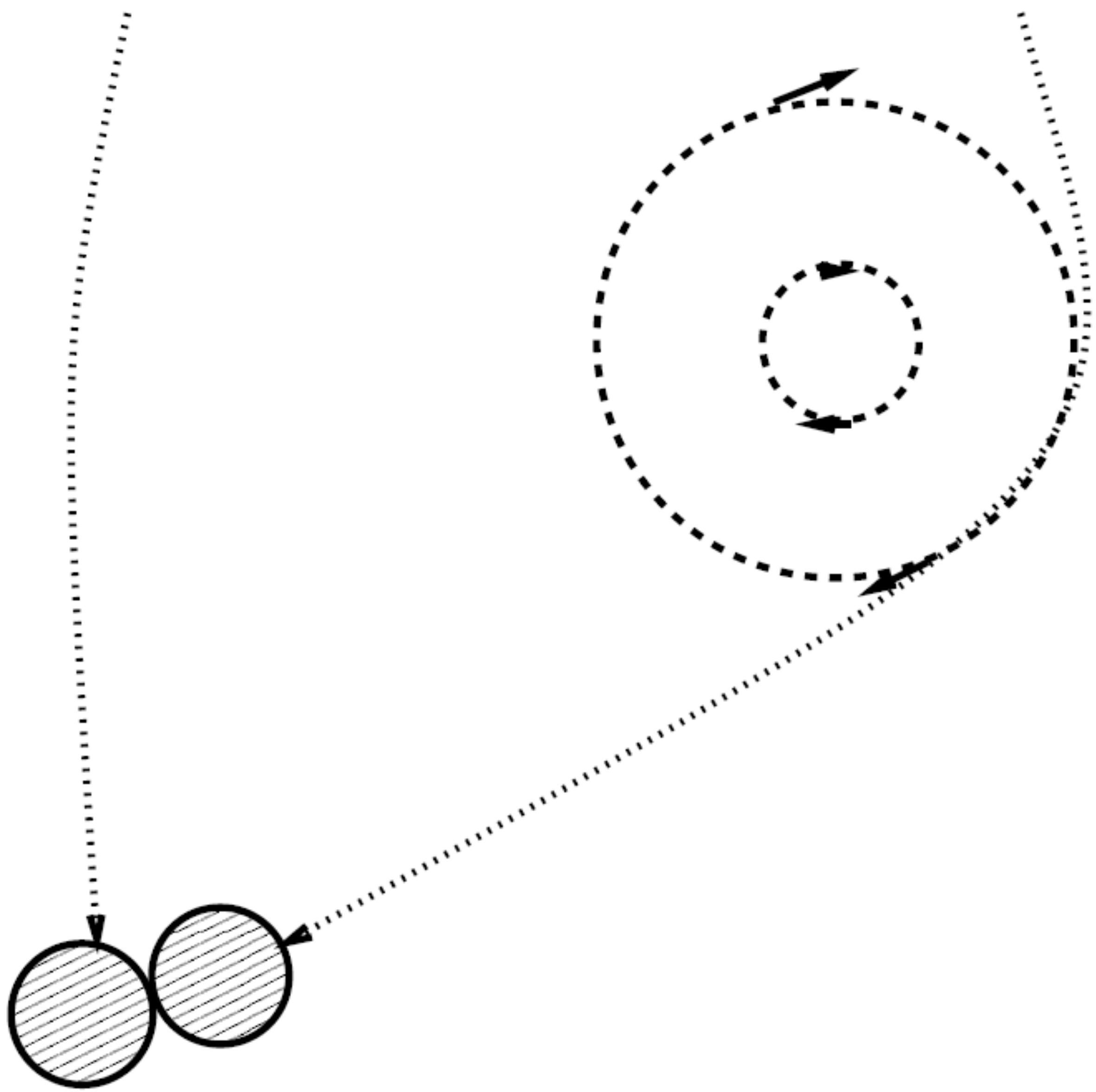
See Cencini lectures

Increase of settling velocity in turbulence



Inertial effects: Preferential concentration and Sling effect

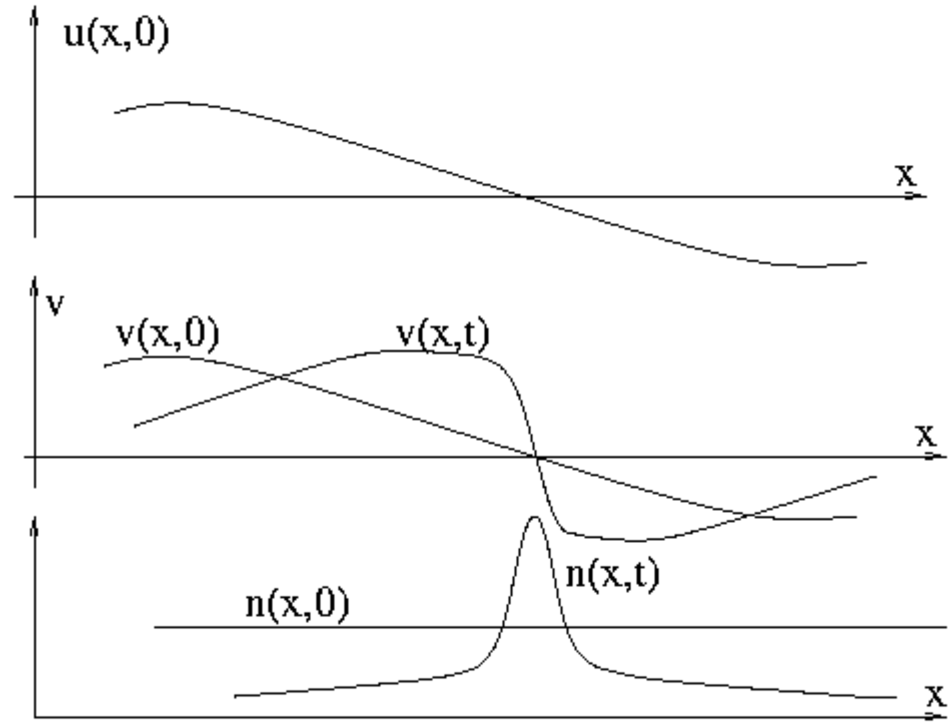




Sling effect: distant vortex causes collisions of droplets

Sling effect and caustics

$$\sigma_{ii} = \partial v_i / \partial x_i < -\tau^{-1} \Rightarrow \sigma_{ii} = (t_0 - t)^{-1} \propto n(q,t)$$



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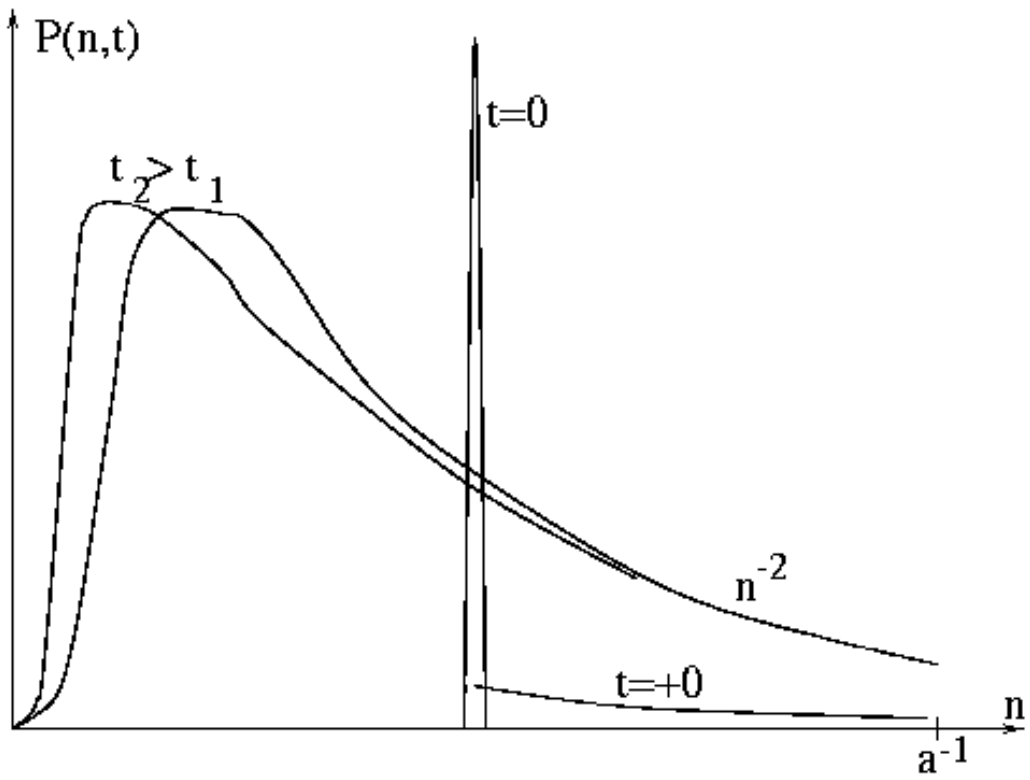
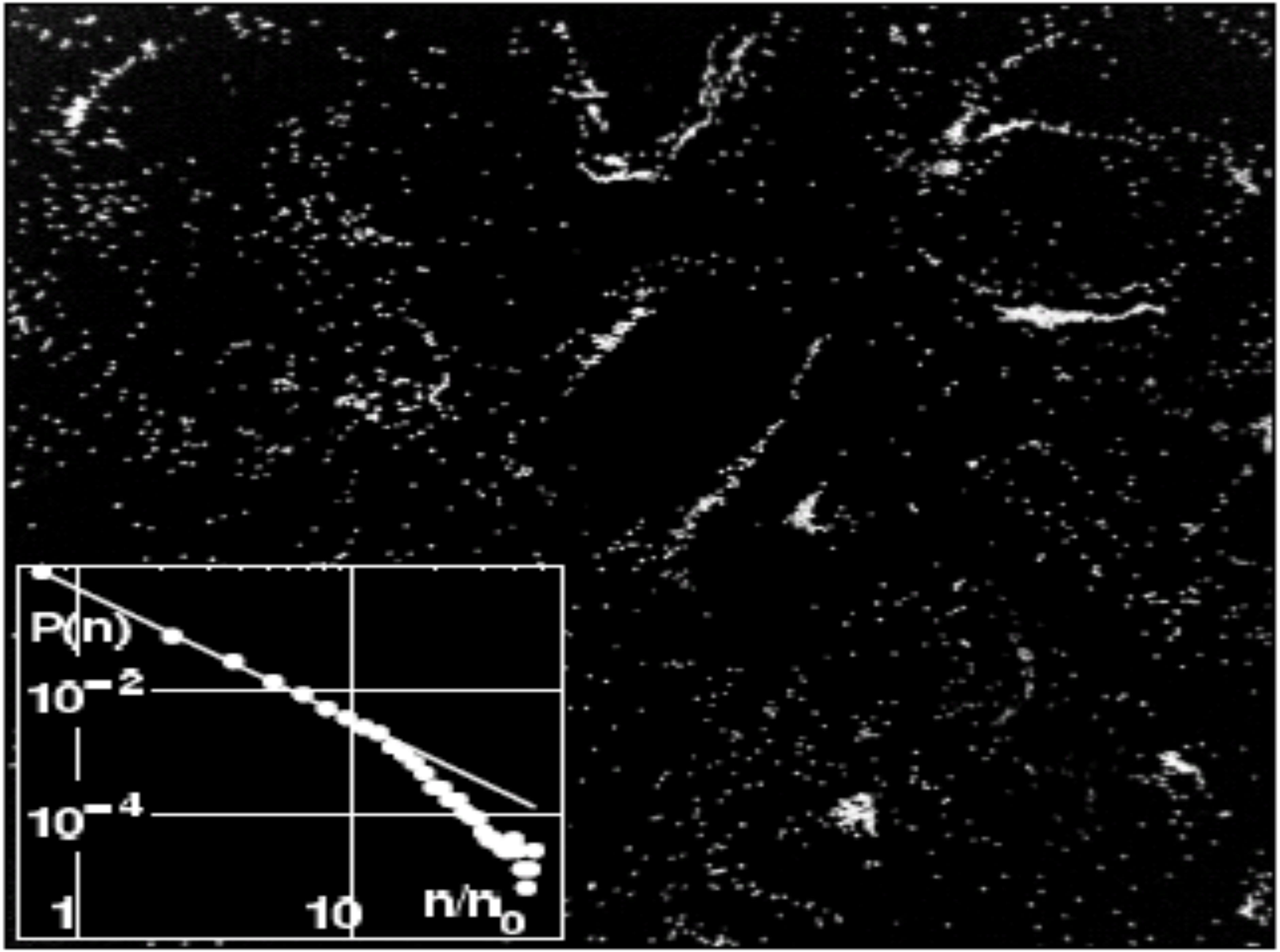
$$n(t) \propto 1/(t_0 - t) \Rightarrow P(n)dn = dt = n^{-2}dn$$



$$P(n,t) \propto n^{-2} \exp(-C/Dt^3)$$



P(n)
10−2
10−4
1
10
n/n0

Collision rate

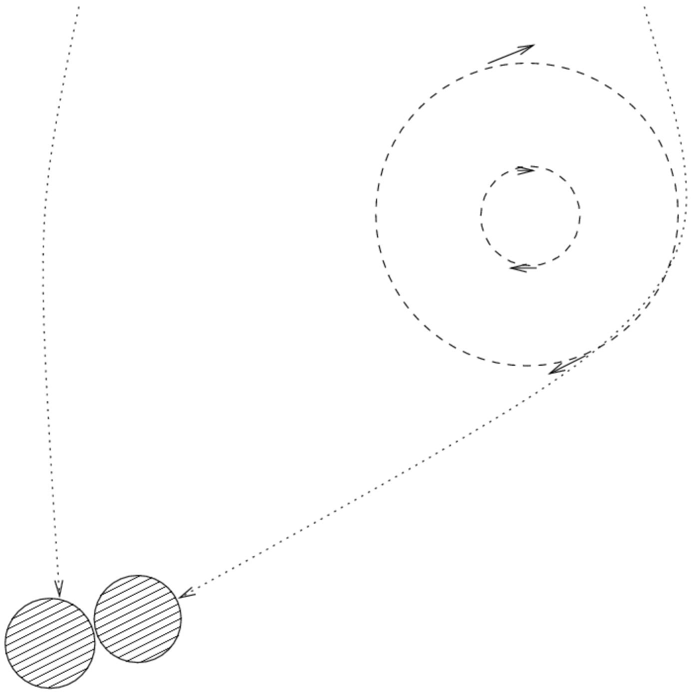
$$K(a,a') = \pi(a+a')^2 \Delta v P(a+a')$$

$$P(l) \sim (\eta/l)^\alpha$$

Sundaram, Collins; Balkovsky, Fouxon, GF

$$K(a,a)/8\pi a^2 = K_{sling} + a\langle \sigma n_a^2 \rangle_E \sim F\lambda\eta + \lambda a \bar{n}^2 (\eta/a)^{2d-\zeta_2}$$

Fouxon, Stepanov, GF



$$St^{-1/2} exp(-A/St)$$

Bezugly, Mehlig and Wilkinson

$$F = St^{-2} \times exp(-A/St) \times (A + BSt^c)$$

Pumir, GF

$$A = 2.1 \text{ for } R_\lambda = 45,$$

$$A = 1.70 \text{ for } R_\lambda = 105$$

turbulent collision kernel

sling effect

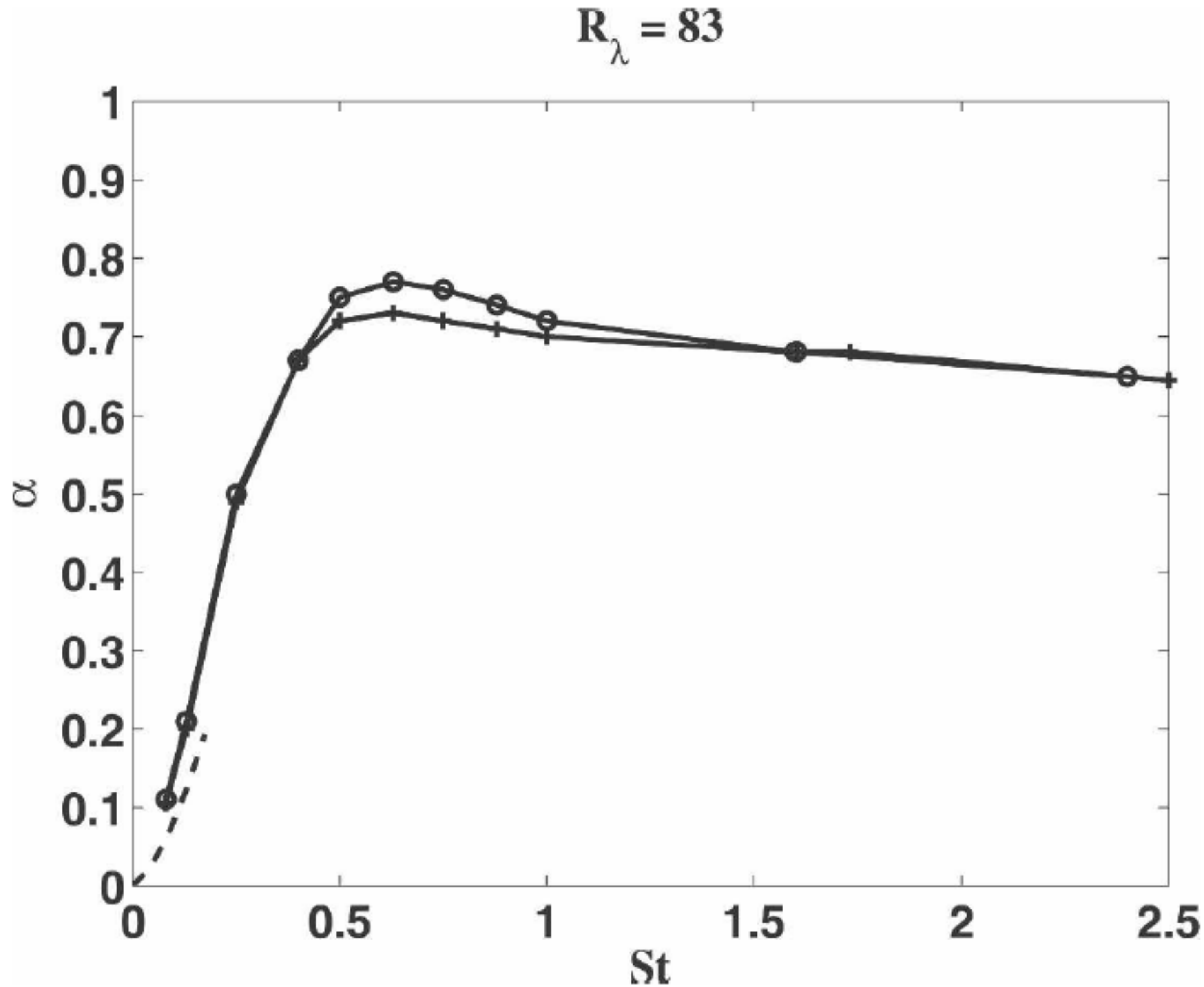
$$K_t = 4\pi\lambda a^3\{(30\pi)^{-1/2} g(a) + 0.3 \exp[-1.7/St(a)]\}$$

preferential concentration

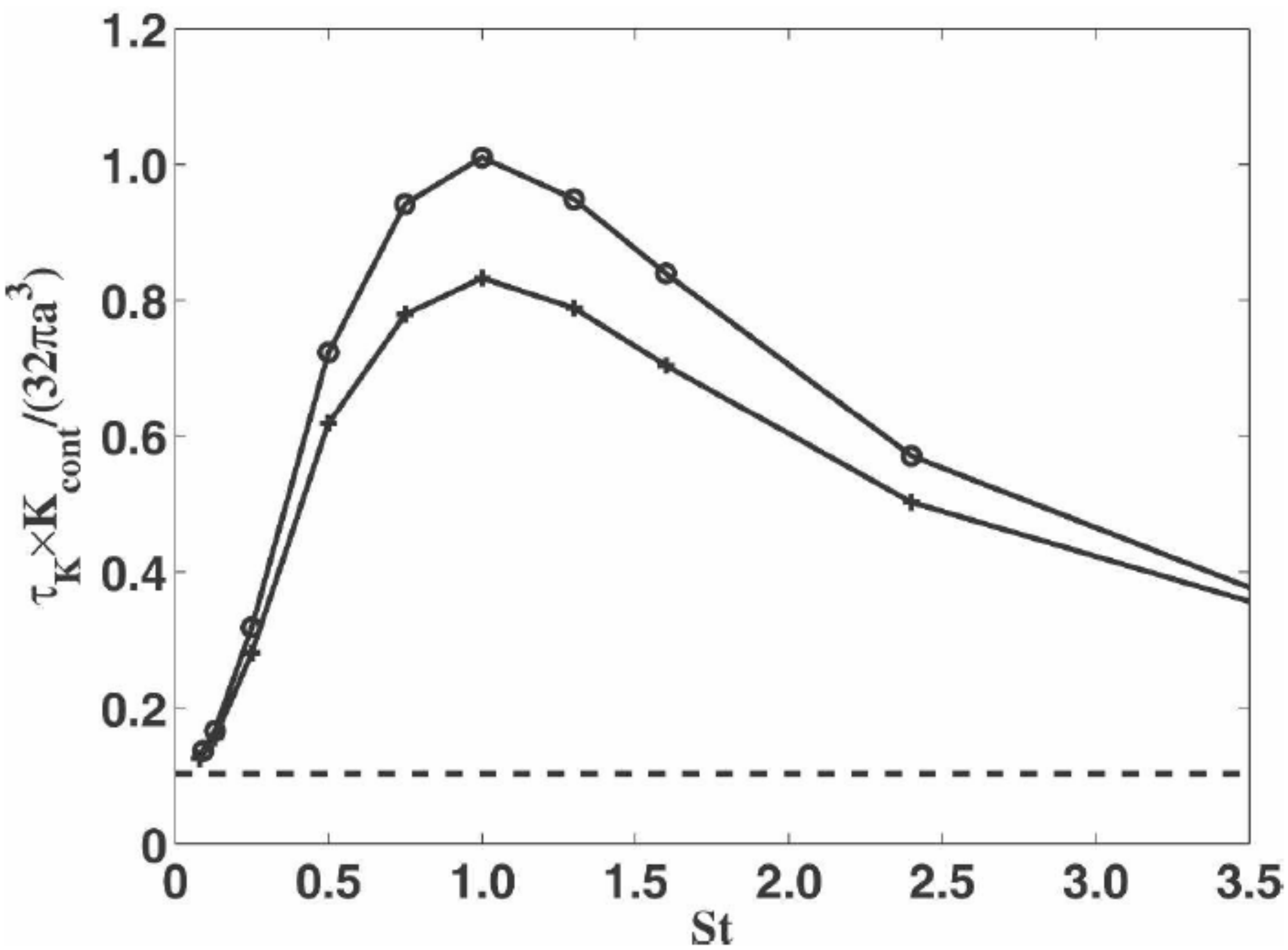
$$g(a) = (\eta/a)^{\alpha}$$

$$\alpha = (3/4)St^2/(0.1 + St^3)$$

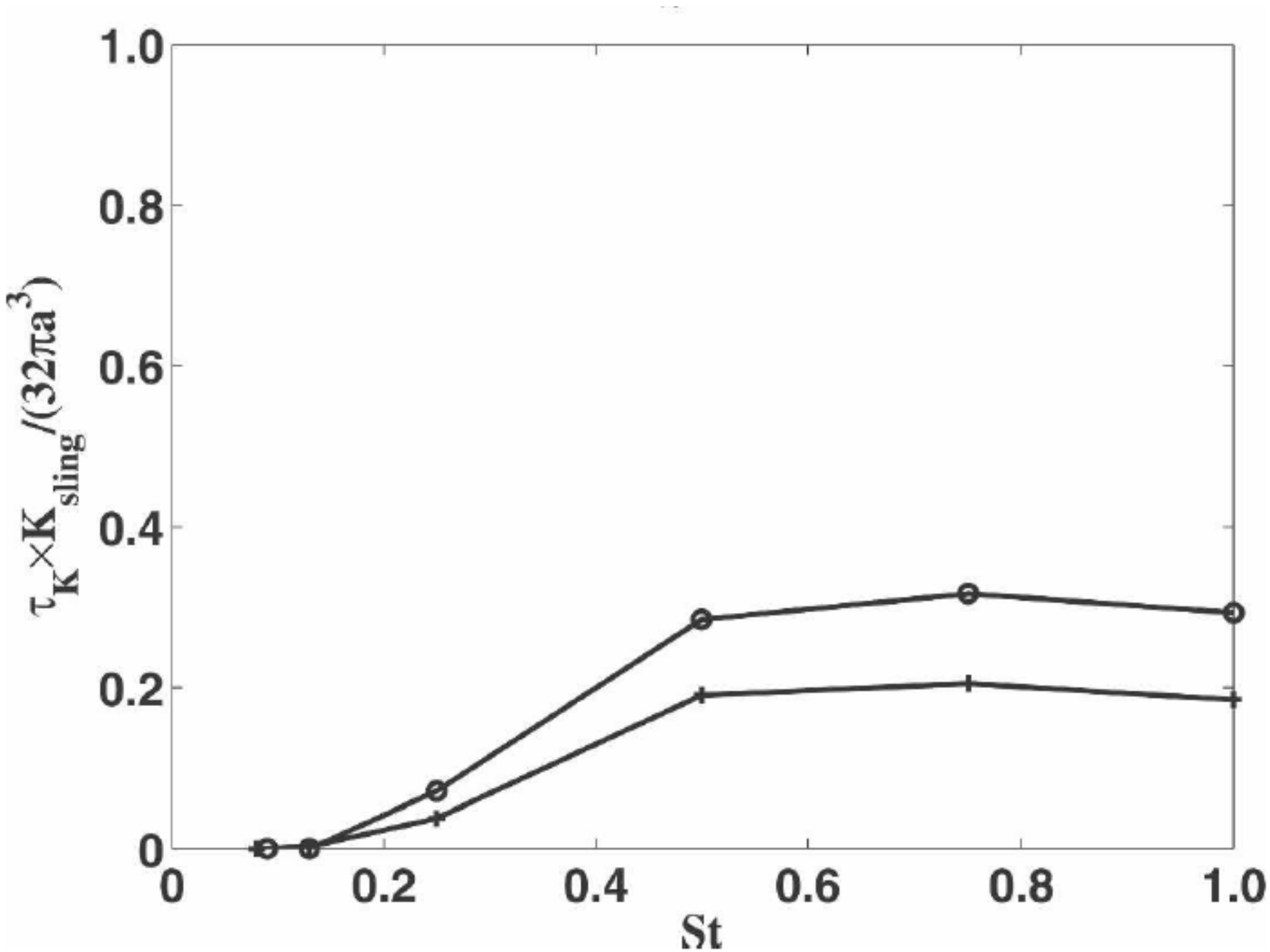
$$g(a) = 1 \text{ when } |a_1 - a_2| > 1\ \mu\text{m}$$


R$_\lambda$ = 83
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
α
0
0.5
1
1.5
2
2.5
St

Continuous contribution



Sling contribution



$$\boxed{\begin{aligned}\frac{\partial n}{\partial t} - \operatorname{div} D(r)\nabla n = & -\frac{\kappa s M}{\rho_0}\frac{\partial}{\partial a}\frac{n(a)}{a} - n(a)\frac{u_{\mathrm{g}}(a)}{L} \\ & + \int \mathrm{d}a' \left[\frac{K(a', a'')n(a')n(a'')}{2(a''/a)^2} - K(a', a)n(a')n(a)\right], \\ \frac{\partial M}{\partial t} - \operatorname{div} D(r)\nabla M = & -4\pi s M\kappa \int a n(a)\,\mathrm{d}a + S.\end{aligned}}$$

supersaturation fluctuations

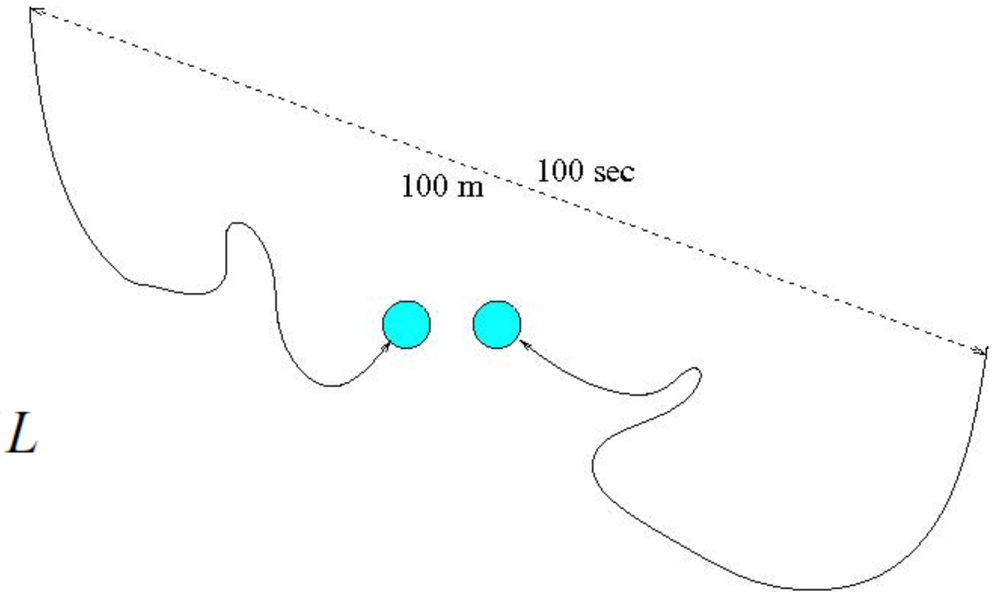
two points a distance r apart separate to the viscous scale during the time $\underline{\lambda^{-1}\ln(\eta/r)}$

then to the distance R during $\underline{(R^2/\epsilon)^{1/3}}$

For $\epsilon = 100\,\mathrm{cm}^2\,\mathrm{s}^{-3}$

separating from $r = 10\,\mu\mathrm{m}$ to

$R = 100\,\mathrm{m}$ takes on average 100 sec



$S = wsM/L$ or wsM_0/L

Typical timescales

inverse droplet growth time, $\kappa M/\rho_0 a^2 \simeq 10^{-2}\,\mathrm{s}^{-1}$

vapour depletion rate

$4\pi\kappa a n \sim 12 \times 0.25\,\mathrm{cm}^2\,\mathrm{s}^{-1} \times 50\,\mathrm{cm}^{-3} \times 10^{-3}\,\mathrm{cm} \simeq 0.15\,\mathrm{s}^{-1}$

inverse turnover time $\quad \epsilon = 100\,\mathrm{cm}^2\,\mathrm{s}^{-3}$

$$w/L \simeq (\epsilon/L^2)^{1/3} \leqslant 10^{-3}\,\mathrm{s}^{-1}$$

collision rate $Kn \simeq 10^{-4} \div 10^{-2}\,\mathrm{s}^{-1}$