THE EQUATIONS OF EXTENDED MAGNETOHYDRODYNAMICS

NICOLAS BESSE AND CHRISTOPHE CHEVERRY

Abstract. Extended magnetohydrodynamics (XMHD) is a fluid plasma model [17] generalizing ideal MHD by taking into account the impact of Hall drift effects [28] and the influence of electron inertial effects [32]. XMHD has a Hamiltonian structure which has received over the past ten years a great deal of attention among physicists [1, 11, 14, 31, 38], and which is embodied by a non canonical Poisson algebra on an infinite-dimensional phase space. XMHD can alternatively be formulated as a nonlinear evolution equation. Our aim here is to investigate the corresponding Cauchy problem. We consider both incompressible and compressible versions of XMHD with, in the latter case, some additional bulk (fluid) viscosity. In this context, we show that XMHD can be recast as a well-posed symmetric hyperbolic-parabolic system implying pseudo-differential operators of order zero acting as coefficients and source terms. Along these lines, we can solve locally in time the associated initial value problems, with moreover a minimal Sobolev regularity. We also explain the emergence and propagation of inertial waves [3, 38].

Keywords. Hyperbolic-parabolic symmetric systems of conservation laws; Initial value problem for nonlinear systems of PDEs; Partially elliptic systems; Compressible and incompressible fluid mechanics; Plasma physics; Hall, Inertial and Extended Magnetohydrodynamics; Pseudo-differential operators; Weyl quantization.

Table of contents

1. Introduction

Extended magnetohydrodynamics (XMHD in abbreviated form) is a system of nonlinear evolution equations in the $(3+1)$-dimensional spacetime $\mathbb{R}^3_{\mathrm{x}} \times \mathbb{R}_t$, which was first initiated by Lüst [32]. It can be obtained [1, 17] from a two-fluid model (electrons plus ions), under the assumptions of quasi-neutrality and smallness of the electron mass compared to the ion mass, by imposing an auxiliary ordering on the equations of motion; it can be recovered from kinetic theory [22]; or, starting with a Lagrangian picture carried by some adequate two-fluid functional, it can be derived from action principles [11, 26]. XMHD can be formulated as a (non-canonical) Hamiltonian system [1, 14] which subsumes ideal MHD, Hall MHD, as well as inertial MHD models. It is equipped with a non canonical Poisson bracket [14], a conserved energy [27], Casimir invariants and topological properties which are investigated in [18, 24, 31] and references therein.

Extended MHD is motivated by its great importance in astrophysics and geophysics. It has proven to be useful in several contexts, like solar wind [3], neutron stars [4] and nuclear fusion science. In fact, the Hall and electron inertial effects play a significant role in XMHD turbulence[1] [3, 38]; they are currently identified [18, 19] as potential sources of fast magnetic reconnection mechanisms[2]; and they must be taken into account to obtain more reliable reduced models for fusion plasmas [8]. These phenomena are dynamical processes. Hence, the importance of developing the Eulerian approach in parallel to the aforementioned Lagrangian viewpoint. This is precisely the position of this article, namely to explain how the XMHD evolution equation can be solved starting from initial data.

Normalizing variables in the standard Alfvén units, with $\nabla \equiv \nabla_{\mathrm{x}}$, XMHD is built with the continuity equation (on the total mass density ρ and the center-of-mass velocity v),

$$\partial_t \rho + \nabla \cdot (\rho\, \mathrm{v}) = 0\,, \tag{1.1}$$

and with the following equation for the momentum density[1] (where $\mathrm{p} : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function of ρ representing a pressure, B is the magnetic field and $\jmath$ is the current density)

$$\rho\, \big(\partial_t \mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v}\big) + \nabla \mathrm{p} - \jmath \times \mathrm{B} + d_e^2\, (\jmath \cdot \nabla)(\jmath/\rho) = 0\,, \tag{1.2}$$

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[1] Let $w_\star := \mathrm{v}_\star - \mathrm{v}$ be the velocity of species $\star$ related to the center of mass velocity v. In a two-fluid context, the net fluid momentum equation involves the momentum flux term $\rho_i\, \mathrm{v}_i \otimes \mathrm{v}_i + \rho_e\, \mathrm{v}_e \otimes \mathrm{v}_e$ which can be decomposed in the form $\rho\, \mathrm{v}\otimes\mathrm{v} + \mathbb{P}_{\mathrm{d}}$ where $\mathbb{P}_{\mathrm{d}} := \rho_i\, w_i \otimes w_i + \rho_e\, w_e \otimes w_e$ is the diffusion pressure tensor which is a source of XMHD plasma turbulence [3, 38]. Under quasi-neutrality, this contribution $\mathbb{P}_{\mathrm{d}}$ gives rise to the contribution $d_e^2\, (\jmath \cdot \nabla)(\jmath/\rho)$ inside (1.2). For more details, see the plasma notes by E. Alec Johnson: *A derivation of Ohm's law and extended MHD starting from two-fluid equations*. One of our conclusions is that the influence of $\mathbb{P}_{\mathrm{d}}$ can be incorporated through the propagation of inertial waves.

[2] In ideal and Hall MHD, the magnetic field undergoes a Lie advection. It follows that the field lines are conserved (this is Alfvén's theorem) and that reconnection cannot occur. By contrast, the non-ideal electric field in the generalized Ohm's law (1.4) can break the frozen-in flux condition. XMHD equations work at scales which are amenable to collisionless magnetic reconnection as confirmed by theoretical considerations [1, 2] and especially by observational evidence [29] in space and astrophysical plasmas (e.g., solar flares and magnetospheres). In fusion devices, two-fluid effects must be considered to explain the fast magnetic reconnection dynamics [21], which is identified as a process responsible for sawtooth oscillations or crashes (which may cause a loss of heat and fast particles, or even a disruption).

The equations (1.1) and (1.2) must be completed with the Maxwell-Ampère equation $\jmath = \nabla \times \mathrm{B}$ (where the displacement current $\partial_t \mathrm{E}$ is dropped under the assumption that our system is not relativistic), with the Maxwell–Faraday equation

$$\partial_t \mathrm{B} + \nabla \times \mathrm{E} = 0\,, \tag{1.3}$$

and with a generalized Ohm's law [27], which gives the electric field E in terms of the other unknowns ρ, v, B, $\jmath$ and the electron pressure p_e according to (see [1, 18])

$$\begin{aligned} \mathrm{E} + \mathrm{v} \times \mathrm{B} = & -\frac{d_i}{\rho}\ \nabla \mathrm{p}_e + d_i\ \frac{\jmath}{\rho} \times \mathrm{B} - d_i\ d_e^2\ \Big(\frac{\jmath}{\rho}\cdot\nabla\Big)\frac{\jmath}{\rho} \\ & + d_e^2\ \Big[\partial_t\Big(\frac{\jmath}{\rho}\Big) + (\mathrm{v}\cdot\nabla)\Big(\frac{\jmath}{\rho}\Big) + \Big(\frac{\jmath}{\rho}\cdot\nabla\Big)\mathrm{v}\Big]\,. \end{aligned} \tag{1.4}$$

The above two dimensionless parameters d_e and d_i are independent. They are non-negative ($d_e \geq 0$ and $d_i \geq 0$). They represent respectively the normalized electron and ion skin depths. In practice (see Remark 5), they are often found to be adjusted in such a way that $0 \leq d_e \leq d_i \ll 1$. Knowing that, the relation (1.4) appears clearly as a perturbation of the ideal Ohm's law ($\mathrm{E} + \mathrm{v} \times \mathrm{B} = 0$).

In Subsection 1.1, we recall the state of knowledge concerning the mathematical results about ideal, Hall and extended MHD. In Subsections 1.2 and 1.3, we present our main outcomes concerning respectively the incompressible and compressible frameworks. This is also an occasion to outline the plan of the text and to emphasize some key ideas.

1.1. **Mathematical background.** We can replace $\jmath$ inside (1.2) and (1.4) by $\jmath = \nabla \times \mathrm{B}$. We can substitute the electric field E thus obtained through (1.4) at the level of (1.3). When doing this, we can collect the quantities which undergo a time derivative, namely

$$\partial_t \mathrm{B} + d_e^2\ \nabla \times \partial_t\Big(\frac{\jmath}{\rho}\Big) = \partial_t\Big[\mathrm{B} + d_e^2\ \nabla \times \Big(\frac{\nabla \times \mathrm{B}}{\rho}\Big)\Big]\,.$$

By this way, the expression

$$\mathrm{B}^* = \mathrm{B} + d_e^2\ \nabla \times \left(\frac{\nabla \times \mathrm{B}}{\rho}\right). \tag{1.5}$$

acquires the status of a dynamical variable. The notation B^* is very common [1, 17, 31]. From there, the unknowns are ρ, v and B^*, while the *constitutive relation* (1.5) is aimed to express B in terms of B^*. After some calculations, or see directly the three equations (1)-(5)-(6) in [31], we obtain the version of XMHD which is highlighted in [1, 14, 31] and which is delivered in the form

$$\left\{ \begin{aligned} & \partial_t \rho + (\mathrm{v}\cdot\nabla)\rho + \rho\ \nabla\cdot\mathrm{v} = 0\,, \\ & \partial_t \mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v} + \frac{\nabla \mathrm{p}}{\rho} + \mathrm{B}^* \times \frac{\nabla \times \mathrm{B}}{\rho} + d_e^2\ \nabla\Big(\frac{|\nabla \times \mathrm{B}|^2}{2\,\rho^2}\Big) = \nu\ d_e^2\ \nabla(\nabla\cdot\mathrm{v})\,, \\ & \partial_t \mathrm{B}^* + \nabla \times \Big(\mathrm{B}^* \times \Big(\mathrm{v} - d_i\ \frac{\nabla \times \mathrm{B}}{\rho}\Big)\Big) + d_e^2\ \nabla \times \Big((\nabla \times \mathrm{v}) \times \frac{\nabla \times \mathrm{B}}{\rho}\Big) = 0\,. \end{aligned} \right. \tag{1.6}$$

In physics textbooks, these equations are often supplemented by

$$\nabla \cdot \mathrm{B}^* = 0\,, \qquad \nabla \cdot \mathrm{B} = 0\,. \tag{1.7}$$

The equations (1.5) and (1.6) are derived (for $\nu = 0$) in the contributions [1, 3, 4, 14, 27, 31] which mainly focus on the Hamiltonian formalism while the Eulerian approach is not really addressed. Individually, the equation (1.6) does not fall into usual mathematical categories and its well-posedness does not appear to have been clarified. In fact, the system (1.5)-(1.6) looks like a quasilinear equation with various second order terms whose different roles need to be identified. The part $\nu\ d_e^2\ \nabla(\nabla \cdot \mathrm{v})$ where $\nu > 0$ represents a bulk (fluid) viscosity. It clearly provides some partial ellipticity on the component v, namely a control on $\nabla \cdot \mathrm{v}$. But the other (nonlinear) second order terms (which are driven by $d_e \geq 0$ and $d_i \geq 0$) do not. Let us consider what can be said about the influence of d_e and d_i.

First, assume that $d_e = 0$. Then, from (1.5), we deduce that $\mathrm{B}^* = \mathrm{B}$, and two situations may be distinguished. For $d_i = 0$, we recover the equations of compressible MHD [33]. For $d_i > 0$, we incorporate the *Hall current term* coming from the third equation of (1.6) which (for $\rho \equiv 1$) reduces to $d_i\, \nabla \times \big((\nabla \times \mathrm{B}) \times \mathrm{B}\big)$. In particular, when $\rho \equiv 1$ and $\nabla \cdot \mathrm{v} = 0$, we find the incompressible Hall-MHD system introduced by Lighthill [28].

The situation $d_e = 0$ has been much studied by mathematicians in recent years, see for instance [9, 13, 30, 44]. This has been achieved in the presence of dissipative terms, namely a shear (fluid) viscosity ($\mu\, \Delta \mathrm{v}$ with $\mu > 0$) and/or a magnetic resistivity ($\eta\, \Delta \mathrm{B}$ with $\eta > 0$). As soon as $\eta > 0$, the second order terms with d_i in factor can be absorbed, and the system becomes locally well-posed. But when $\eta = 0$, Hall-MHD equations are today known to be strongly ill-posed [10], even in Gevrey spaces [23], and even if some kinematic viscosity $\mu\, \Delta \mathrm{v}$ with $\mu > 0$ is added.

In this text, as prescribed by physicists [1, 14, 27, 31, 32], we work with $d_e > 0$. This passage from $d_e = 0$ to $d_e > 0$ is very significant since it allows to include *inertial effects* that are fundamental in plasma dynamics[3]. We make progress in two directions:

• Looking at the content of (1.6), this improvement (from $d_e = 0$ to $d_e > 0$) is already quite an achievement. Indeed, the situation $d_e > 0$ seems more complicated: the symmetric part of ideal MHD is broken (since B is substituted for B^*); the Hall term (with its potential instabilities [10, 23]) is still present; and there are extra nonlinear second order terms without evident sign conditions. Clearly, supplementary derivative losses may be expected, while the introduction of d_e does not furnish any dissipation. That is probably why the Cauchy problem associated with (1.6) has not yet (to our knowledge) been investigated.

• In line with the preceding mathematical approaches, we use a touch of dissipation. We impose a bulk (fluid) viscosity $\nu\, d_e^2 > 0$. This condition is not demanding. In particular, it disappears when the flow is incompressible. The key highlight is, unlike [9, 13, 30, 44], the absence of shear (fluid) viscosity ($\mu\, \Delta \mathrm{v}$ with $\mu > 0$) and magnetic resistivity ($\eta\, \Delta \mathrm{B}$ with $\eta > 0$). This means that the Hall instabilities [10, 23] can (locally in time) be compensated by inertial effects ($d_e > 0$) without resorting to such additional dissipative terms.

[3] See Remark 5 which explains (by looking at plasma parameters) why $d_e > 0$.

The question is why ? Our claim is that (1.6) becomes locally well-posed once $d_e > 0$ and (in the compressible case) once $\nu > 0$ for the following two principal reasons:

- *About the influence of* $d_e > 0$. The analysis of derivative losses (when $d_e = 0$) does not include the constitutive relation (1.5) doing everything completely differently by modifying the role of B from $\mathrm{B} \equiv \mathrm{B}^*$ (when $d_e = 0$) to some another $\mathrm{B} \not\equiv \mathrm{B}^*$ (when $d_e > 0$) with a gain of derivatives. Despite appearances, by a change of unknowns, the system (1.6) can be recast as a (foliation of) well-posed hyperbolic-parabolic systems (whose coefficients and source terms take the form of zero order pseudo-differential operators). In so doing, the inertial terms (those with $d_e > 0$ in factor) contribute to some (almost) symmetric structure, involving completely new features. In this interpretation, they do not provide second order dissipative perturbations. Instead, they contribute to the appearance of inertial waves.
- *About the influence of* $\nu > 0$. The introduction of a volume (fluid) viscosity (in place of a magnetic resistivity) is sufficient (and seems also necessary as in other contexts [37]) to absorb (for reasons like in [25]) the problematic contributions that remain in the compressible framework, when performing energy estimates.

As a consequence:

- We will involve changes of variables that become singular when $d_e \in \mathbb{R}_+^*$ goes to zero. Throughout the text, it is therefore essential to work with $d_e > 0$. Keep in mind that there is no smooth passage from the case $d_e > 0$ to the case $d_e = 0$. For instance, in (1.5), B is expressed in terms of B^* through a (partially) elliptic operator which will prove to be (on some appropriate subspace) of order -2 when $d_e > 0$, and of order 0 when $d_e = 0$.
- It is essential to assume that $\nu > 0$ when dealing with the compressible framework.

Knowing that $d_e > 0$, we can prefer a rescaled version of (1.6) that makes us forget the role of d_e. To this end, we multiply x, v, B^* and B by d_e^{-1}, while p is multiplied by d_e^{-2}. In other words, we work with

$$d := \frac{d_i}{d_e}\,, \qquad x := \frac{\mathrm{x}}{d_e}\,, \qquad v := \frac{\mathrm{v}}{d_e}\,, \qquad B^* := \frac{\mathrm{B}^*}{d_e}\,, \qquad B := \frac{\mathrm{B}}{d_e}\,, \qquad p := \frac{\mathrm{p}}{d_e^2}\,.$$

With these conventions, we find

$$\begin{cases} \partial_t \rho + (v\cdot\nabla)\rho + \rho\ \nabla\cdot v = 0\,, \\ \partial_t v + (v\cdot\nabla) v + \dfrac{\nabla p}{\rho} + B^*\times\dfrac{\nabla\times B}{\rho} + \nabla\Big(\dfrac{|\nabla\times B|^2}{2\,\rho^2}\Big) = \nu\ \nabla(\nabla\cdot v)\,, \\ \partial_t B^* + \nabla\times\Big(B^*\times\Big(v - d\,\dfrac{\nabla\times B}{\rho}\Big)\Big) + \nabla\times\Big((\nabla\times v)\times\dfrac{\nabla\times B}{\rho}\Big) = 0\,, \end{cases} \tag{1.8}$$

together with

$$B^* = B + \nabla\times\Big(\frac{\nabla\times B}{\rho}\Big)\,. \tag{1.9}$$

Let $\bar\rho \in \mathbb{R}_+^*$ be a constant positive background density. At the initial time $t = 0$, we impose

$$(\rho, v, B^*)(0,\cdot) = (\bar\rho + \rho_0, v_0, B_0^*)\,. \tag{1.10}$$

We work away from vacuum, say with

$$0 < \bar\rho/2 \le \bar\rho + \rho_0(x)\,. \tag{1.11}$$

Note that the parameter d_e is no more visible at the level of (1.8)-(1.9). It is in fact hidden behind the definition of d and behind the preceding change of scales. It keeps of course some influence. Indeed, let (ρ, v, B^*) be a solution to (1.8)-(1.9). We can adjust d_i in such a way that $d_i = d\, d_e$ for a fixed $d \ge 0$, and consider that d_e can vary. Coming back to the initial variables, we find that

$$(\rho, \mathrm{v}, \mathrm{B}^*)(t,\mathrm{x}) := (\rho, d_e\, v, d_e\, B^*)(t, \mathrm{x}/d_e)\,, \qquad d_e \in]0,1]\,, \tag{1.12}$$

is a family of solutions to (1.6) which belongs (when $d_e > 0$ goes to 0) to a perturbative concentrating regime[4] (the periodic regime will not be investigated here) near the constant solution $(\bar\rho, 0, 0)$. Indeed, the velocity and magnetic components (v and B^*) are of small amplitude d_e while the profiles are in $\mathcal{H}^s$ (and thus they are decreasing functions, typically compactly supported). Retain that d_e has a significant impact at the level of (1.6), when looking at $(\rho, \mathrm{v}, \mathrm{B}^*)$. However, d_e will not be apparent in our statements (which are uniform with respect to d_e) since they are formulated in terms of (1.8)-(1.9). As usual, we denote by $\mathcal{F}$ the Fourier transform and, for $s \in \mathbb{R}$, by $\mathcal{H}^s := \mathcal{F}^{-1}\big(\langle\xi\rangle^s\, \mathcal{F}\, L^2\big)$ the standard Sobolev–Bessel potential space (recall that $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$).

1.2. **The incompressible situation.** This entails looking at the pressure $p : \mathbb{R}_+ \to \mathbb{R}$ as a scalar function that plays the role of a Lagrange multiplier. This also requires to start with initial data as in (1.10) with $\rho_0 = 0$ as well as

$$\nabla\cdot v_0 = 0\,, \qquad \nabla\cdot B_0^* = 0\,. \tag{1.13}$$

Equivalently (see Subsection 2.1), the incompressible situation implies that:

i) The density ρ is a positive constant, say (without limiting the generality)

$$\rho = \bar\rho = 1\,. \tag{1.14}$$

ii) All the vector fields v, B^* and B belong to $\mathcal{D}^s := \big\{\, D \in \mathcal{H}^s(\mathbb{R}^3;\mathbb{R}^3)\,;\, \nabla\cdot D = 0 \,\big\}$. In other words, they are solenoidal

$$\nabla\cdot v = 0\,, \qquad \nabla\cdot B^* = 0\,, \qquad \nabla\cdot B = 0\,. \tag{1.15}$$

iii) The set (1.8) of equations reduces to

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p + B^*\times(\nabla\times B) = 0\,, \\ \partial_t B^* + \nabla\times\big(B^*\times(v - d\,\nabla\times B)\big) + \nabla\times\big((\nabla\times v)\times(\nabla\times B)\big) = 0\,. \end{cases} \tag{1.16}$$

iv) The constitutive relation is replaced by

$$B = (\mathrm{Id} - \Delta)^{-1} B^*\,. \tag{1.17}$$

[4]The regime of geometric optics is weakly nonlinear with *short pulses* [7] at hand.

Theorem 1. *[Local smooth wellposedness for incompressible XMHD] Fix the initial data such that*

$$(v,B^*)(0,\cdot)=(v_0,B_0^*)\in\mathcal{D}^s(\mathbb{R}^3;\mathbb{R}^3)\times\mathcal{D}^{s-1}(\mathbb{R}^3;\mathbb{R}^3)\,,\qquad s>5/2\,. \tag{1.18}$$

Then, we can find some time $T>0$ depending only on the $\mathcal{H}^s\times\mathcal{H}^{s-1}$-norm of (v_0,B_0^*) such that the Cauchy problem built with (1.15)-(1.16)-(1.17) together with the initial condition (1.18) has a unique local solution on $[0,T]$, which is smooth in the following sense

$$(v,B^*,B)\in C\big([0,T];\mathcal{D}^s(\mathbb{R}^3;\mathbb{R}^3)\times\mathcal{D}^{s-1}(\mathbb{R}^3;\mathbb{R}^3)\times\mathcal{D}^{s+1}(\mathbb{R}^3;\mathbb{R}^3)\big)\,. \tag{1.19}$$

In (1.19), the level $\mathcal{H}^s$ of regularity for v does not match with the one $\mathcal{H}^{s-1}$ obtained for the dynamical variable B^*. This is because the presentation (1.16), though inherited from physics, is not suitable from the perspective of initial value problems. This explains probably why things have not yet worked in this way. To remedy this, we transform in Section 2 the equations of (1.16). More precisely, we incorporate a new equation on the vorticity $w:=\nabla\times v$ in order to obtain a system on (w,B^*) called the *vorticity formulation*. Then, we derive energy estimates up to the proof of Theorem 1 (for $s>7/2$).

1.3. **The compressible framework.** We present below our result concerning (1.8)-(1.9). As a prerequisite, we assume a barotropic equation of state. In other words, the pressure p is only a function of the density ρ. It is prescribed by a smooth given function $p:\mathbb{R}_+\to\mathbb{R}$ whose derivative p' is positive.

Theorem 2 (Local smooth wellposedness for compressible XMHD)**.** *Assume that $\nu>0$, and fix any $s>5/2$. Select some initial data as in (1.10)-(1.11), with moreover*

$$(\rho_0,v_0,B_0^*)\in\mathcal{H}^s(\mathbb{R}^3;\mathbb{R})\times\mathcal{H}^s(\mathbb{R}^3;\mathbb{R}^3)\times\mathcal{H}^{s-1}(\mathbb{R}^3;\mathbb{R}^3)\,,\qquad \nabla\cdot B_0^*\in\mathcal{H}^s(\mathbb{R}^3;\mathbb{R}^3)\,. \tag{1.20}$$

Then, we can find some time $T>0$ which is proportional to the parameter ν and inversely proportional to the $\mathcal{H}^s\times\mathcal{H}^s\times\mathcal{H}^{s-1}\times\mathcal{H}^s$-norm of $(\rho_0,v_0,B_0^*,\nabla\cdot B_0^*)$ such that the Cauchy problem built with (1.8)-(1.9) together with (1.10)-(1.11)-(1.20) has a unique local solution on $[0,T]$, which is smooth in the following sense

$$(\rho,v,B^*,B)\in C\big([0,T];\mathcal{H}^s(\mathbb{R}^3;\mathbb{R})\times\mathcal{H}^s(\mathbb{R}^3;\mathbb{R}^3)\times\mathcal{H}^{s-1}(\mathbb{R}^3;\mathbb{R}^3)\times\mathcal{H}^{s+1}(\mathbb{R}^3;\mathbb{R}^3)\big)\,. \tag{1.21}$$

Let us suppose, as it is often the case in practice, that $0<d_e\le d_i\ll 1$. Then, our analysis indicates that the system (1.6) involves a mix of three interconnected regimes:

- For low frequencies $|\xi|\lesssim 1$, the solutions behave (approximately) as provided for by compressible magnetohydrodynamics [5, 33, 41].
- For intermediate frequencies $d_i^{-1}\lesssim|\xi|\ll d_e^{-1}$, the Hall effects come into play [9, 13, 30, 28, 44], and various amplification mechanisms become to be implemented. This includes a step towards the singularity formations detected by mathematicians [10, 23] and the tearing modes studied by physicists [16, 18] in the perspective of collisionless magnetic reconnection. However, in the weakly nonlinear regime (1.12) and as long as the time remains finite, these instabilities do not induce the explosion (of norms) and they do not jeopardize the construction of solutions.

- For large frequencies $d_e^{-1} \lesssim |\xi|$, inertial aspects take the place and new speeds (modes) of propagation appear. This means the emergence of *inertial waves* (see Paragraph 2.3.2), whose impacts have been already observed by physicists [3, 38] but which do not seem to have been mathematically well identified before.

It should be borne in mind that Theorem 1 is more accessible than Theorem 2. To some extent, it can be viewed as a simplified version of it. This is why the analysis begins in Section 2 with completing the incompressible situation. This makes the basic ideas more accessible. This also furnishes clear guidelines in the perspective of the compressible framework which is investigated in Section 3.

Section 3 follows the same steps as in Section 2 but it faces new challenges:

- On the one hand, in comparison with (1.17), due to the variations of ρ, it is more difficult to exhibit the properties of (partial) ellipticity which are hidden behind the constitutive relation (1.9), see Subsection 3.1.
- On the other hand, the incompressible transformation must be adapted to the compressible framework, see Subsection 3.2. We still add the vorticity $w = \nabla \times v$. Besides, we implement the divergence $\nabla \cdot v$ and one order derivatives of ρ. The system thus obtained is called the *compressible vorticity formulation*.
- In subsection 3.3, we remark that the divergence of B^* is a preserved quantity. Taking advantage of this information, we show that there is no loss of hyperbolicity and that energy estimates become available. This is the entry point to the proof of Theorem 2 (at least for $s > 7/2$).

Another salient point should be reported. When dealing in space dimension $d = 3$ with Sobolev solutions to quasilinear systems, the restriction $s - 1 > 1 + (d/2) = 5/2$ (or equivalently $s > 7/2$) on the component B^* would be expected [33, 41]. In Theorems 1 and 2, observe the presence of the relaxed condition $B^* \in \mathcal{H}^{(3/2)+}$ instead of the usual constraint $B^* \in \mathcal{H}^{(5/2)+}$. There is a gain of one degree of regularity which is justified in Section 4. To this end, instead of looking at derivatives of (ρ, v), we integrate the magnetic field B^*. As a matter of fact, we consider the magnetic potential A^* which is such that $\nabla \times A^* = B^*$ and $\nabla \cdot A^* = 0$. This leads to the *potential formulation*.

The potential formulation furnishes a self-contained system on (ρ, v, A^*), which can be studied independently and which furnishes different types of supplementary information. This corresponds to the most completed approach but also in some aspects to the most challenging. This is why it is explained lastly. In fact, the potential formulation falls (modulo adaptations) under the scope of Kawashima-Shizuta theory [25]. This leads to the optimal regularity results (with $s > 5/2$) stated in Theorems 1 and 2.

Section 5 is to exhibit the various types of inertial waves that can arise, and to study their properties. To this end, we first select special solutions (constant, in the form of Beltrami fields, corresponding to null point configurations, two dimensional, or even moving). Then, we look at the associated linearized equations and we focus on the regime of high frequencies (with $d_e^{-1} \lesssim |\xi|$). By this way, we can highlight the presence of *inertial dispersion relations* which are of particular interest.

There is a short Appendix, in Section 6. It is about the div-curl system which appears repeatedly throughout the text.

Given a state variable U, we often employ the notation $U^\star_\diamond$. The superscript $\star \in \{\mathfrak{i}, \mathfrak{c}\}$ is to indicate that U is related respectively to the incompressible and compressible situations. The subscript $\diamond \in \{\mathfrak{v}, \mathfrak{p}\}$ (where $\mathfrak{v}$ and $\mathfrak{p}$ must not be confused with the velocity v and the pressure p) refers to the vorticity and potential formulations. We reserve the *rsfs* font $\mathscr{P}$ for operators, with a symbol denoted by the standard font P, so that $\mathscr{P} = P(D_x)$. We often put the subscript $* \in \mathbb{Z}$ to specify that $\mathscr{P}^\star_* = P^\star_*(D_x)$ is of (maximal) order $*$, while the superscript $\star \in \{\mathfrak{i}, \mathfrak{c}\}$ may still be incorporated for the same reasons as before.

2. The incompressible situation

In Subsection 2.1, we introduce the incompressible equations and some of its principal features. In Subsection 2.2, we exhibit properties of ellipticity lying behind (1.17). In Subsection 2.3, we perform a dependent change of unknowns which transforms (1.16). In Subsection 2.4, we derive energy estimates in order to show Theorem 1.

2.1. **The incompressible equations.**

The incompressible situation is strongly linked to the system (1.8)-(1.9) of origin. To see how, starting from (1.8)-(1.9), we have to deduce (1.14), (1.15), (1.16) and (1.17). To this end, we consider below successively the indents i), $\cdots$, iv) of Subsection 1.2.

i) Since $\nabla \cdot v = 0$, the first equation of (1.8) implies that the density ρ is just advected along the characteristic curves generated by the vector field v. Hence it remains constant, say $\rho = \bar\rho = 1$, if initially $\rho_0 = 0$.

ii) As already explained, the term ∇p plays the role of a Lagrange multiplier which ensures the propagation of the constraint $\nabla \cdot v = 0$. On the other hand, it is clear that the divergence-free condition imposed inside (1.13) on B^* at time $t = 0$ is propagated via the divergence of the third equation of (1.8), and that it is transmitted to B through (1.9).

iii) We can always incorporate the part $|\nabla \times B|^2/2\,\rho^2$ to the function p. Then, knowing that $\rho = 1$ and $\nabla \cdot v = 0$, the system (1.8) is exactly the same as (1.16).

iv) The link between B and B^* is here simplified into $B^* = B + \nabla \times (\nabla \times B)$. Now, since $\nabla \cdot B = 0$, we have $\nabla \times (\nabla \times B) = -\Delta B$. After inversion, this yields (1.17).

Before proceeding, we remark that there is a conserved quantity which may be expressed in terms of (v, B).

Lemma 3. *[A conserved quantity] incompressible XMHD preserves the energy*

$$\mathcal{E}^{\mathfrak{i}} := \frac{1}{2} \int_{\mathbb{R}^3} \big(|v|^2 + |B|^2 + |\nabla \times B|^2\big)\, dx\,. \tag{2.1}$$

Proof. Take the L^2-scalar product of the first equation of (1.16) with v. Using integration by parts and the condition $\nabla \cdot v = 0$, all terms vanish except $B^* \times (\nabla \times B)$ giving

$$\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{R}^3} |v|^2\, dx\Big) + \int_{\mathbb{R}^3} v \cdot \big(B^* \times (\nabla \times B)\big)\, dx = 0\,. \tag{2.2}$$

Take the L^2-scalar product of the second equation of (1.16) with B (but not B^*). Perform integration by parts (or exploit that the curl operator is self-adjoint), to see that the two triple products vanish. There remains

$$\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{R}^3}\big(|B|^2+|\nabla\times B|^2\big)\,dx\Big)+\int_{\mathbb{R}^3}(\nabla\times B)\cdot(B^*\times v)\,dx=0\,. \tag{2.3}$$

Summing (2.2) and (2.3), we obtain that $d\mathcal{E}^{\rm i}/dt=0$ as expected. □

Remark 4. *[Similarities with Leray–α models] Incompressible XMHD equations may bear some resemblance to Lagrangian averaged (or Leray–α) Euler equations* [20, 35, 36]*, where a parameter α is introduced and represents the spatial scale below which the dymanics are averaged. But if the parameter d_e can be seen (to some extent) as playing the part of α in Lagrangian averaged α–models, its introduction is driven by other considerations related to two-fluid models* [17, 22] *and its handling is completely different.*

We also come back to the introduction of d, and its significance.

Remark 5. *[Comparison of electron and ion skin depths] Let ω_{pi} and ω_{pe} be the ion and electron plasma frequencies. Under quasi-neutrality (when $q_i\,n_i-e\,n_e=0$), the ratio between d_i and d_e can be expressed in terms of plasma parameters according to*

$$d=\frac{d_i}{d_e}=\frac{\omega_{pe}}{\omega_{pi}}\simeq 42,72\;\frac{\sqrt{n_e}}{\sqrt{n_i}}\;\frac{\sqrt{\mu}}{Z}\simeq 42,72\;\frac{\sqrt{\mu}}{\sqrt{Z}}\,,$$

where n_i and n_e are the number densities of ions and electrons, $\mu=m_i/m_p$ is the ion mass (expressed in units of the proton mass), and $Z=q_i/e$ equals to the atomic number. In a plasma composed only of electrons and protons (where $\mu=1$ and $q_i=e$ so that $Z=1$), we find that $d\simeq 40$. For mixtures of ions and electrons (because μ becomes in general much larger due to the presence of neutrons while Z remains small), we may find that $1\ll d$. However, we emphasize that the two parameters d_e and d_i are completely independent. We refer to the paragraph "Inertial MHD" in [31]*-p.2402) for a rapid discussion about typical values of d_i and d_e.*

2.2. The incompressible constitutive relation.

The operator

$$\nabla\times\nabla\times:\mathcal{H}^2(\mathbb{R}^3;\mathbb{R}^3)\to L^2(\mathbb{R}^3;\mathbb{R}^3)$$

is not elliptic of order 2, since it has a nonzero kernel. To avoid this difficulty, it suffices to restrict its action on a suitable subspace.

Lemma 6. *[Underlying ellipticity when passing from B^* to B through the relation (1.17)] The differential operator*

$$\mathscr{L}_2^{\rm i}:=\mathrm{Id}+\nabla\times\nabla\times:\mathcal{D}^s\to\mathcal{D}^{s-2}\,,\qquad s\in\mathbb{R}\,, \tag{2.4}$$

is bijective and elliptic of order 2. Its inverse $(\mathscr{L}_2^{\rm i})^{-1}:\mathcal{D}^{s-2}\to\mathcal{D}^s$ takes the form of a Fourier multiplier which is elliptic of order -2.

Proof. Let $\big(e_1(\xi), e_2(\xi), e_3(\xi)\big)$ be a smooth orthonormal frame on $\mathbb{R}^3\backslash\{0\}$ which is adjusted such that $e_1(\xi) = \xi/|\xi|$. Let $O_0(\xi)$ be the orthogonal matrix whose column vectors are $e_1(\xi)$, $e_2(\xi)$ and $e_3(\xi)$. In other words

$$O_0(\xi) := \big(e_1(\xi), e_2(\xi), e_3(\xi)\big)\,, \qquad e_1(\xi) = \xi/|\xi|\,, \qquad e_i(\xi)\cdot e_j(\xi) = \delta_{ij}\,. \tag{2.5}$$

Since $\xi \times e_1(\xi) = 0$ whereas $\xi \times \xi \times e_j(\xi) = -|\xi|^2\, e_j(\xi)$ for $j \in \{2,3\}$, we have

$$\mathcal{F}\,(\mathrm{Id} + \nabla\times\nabla\times)\,\mathcal{F}^{-1} = O_0(\xi)\, D_2^{\mathrm{i}}(\xi)\, O_0(\xi)^{-1}, \qquad D_2^{\mathrm{i}}(\xi) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle\xi\rangle^2 & 0 \\ 0 & 0 & \langle\xi\rangle^2 \end{pmatrix}. \tag{2.6}$$

In other words, the action of $\mathrm{Id} + \nabla\times\nabla\times$ on the whole space $\mathcal{H}^2(\mathbb{R}^3;\mathbb{R}^3)$ is unitary equivalent through a conjugation by $\mathscr{O}_0 := O_0(D_x)$ to the diagonal operator $\mathscr{D}_2^{\mathrm{i}} := D_2^{\mathrm{i}}(D_x)$. Introduce the L^2-projectors $\mathscr{P}$ and $\mathscr{Q}$, where $\mathscr{Q} := \mathrm{Id} - \mathscr{P}$ and $\mathscr{P} = P(D_x)$ is given by the Leray projector whose matrix valued symbol is given by

$$P(\xi)\, v := \big(e_2(\xi)\cdot v\big)\, e_2(\xi) + \big(e_3(\xi)\cdot v\big)\, e_3(\xi)\,. \tag{2.7}$$

Recall that the operator $\mathscr{L}_2^{\mathrm{i}}$ is defined by the restriction of its action to $\mathcal{D}^s$, while the set $\mathcal{D}^s$ may be characterized by

$$\mathcal{D}^s := \mathscr{P}\,\mathcal{H}^s(\mathbb{R}^3;\mathbb{R}^3)\,. \tag{2.8}$$

This implies that $\mathscr{L}_2^{\mathrm{i}}$ does not see the eigenvalue 1 of $D_2^{\mathrm{i}}(\xi)$. It just acts on the Fourier side according to the multiplier

$$\mathcal{F}\,\mathscr{L}_2^{\mathrm{i}}\,\mathcal{F}^{-1} = \big(\mathcal{F}\,(\mathrm{Id} + \nabla\times\nabla\times)\,\mathcal{F}^{-1}\big)_{|\mathcal{F}\,\mathcal{D}^s} \equiv \langle\xi\rangle^2\,\mathrm{Id}\,, \qquad \mathscr{L}_2^{\mathrm{i}} \equiv \mathrm{Id} - \Delta\,, \tag{2.9}$$

which is bijective and elliptic of order 2. From (2.9), we infer that

$$\mathcal{F}\,(\mathscr{L}_2^{\mathrm{i}})^{-1}\,\mathcal{F}^{-1} = \langle\xi\rangle^{-2}\,\mathrm{Id}\,, \qquad (\mathscr{L}_2^{\mathrm{i}})^{-1} \equiv (\mathrm{Id} - \Delta)^{-1}\,.$$

This clearly confirms that resorting to $(\mathscr{L}_2^{\mathrm{i}})^{-1}$ allows to gain two derivatives. □

With the above convention, we can deduce from (1.17) the incompressible constitutive relation

$$\nabla\times B = \mathscr{K}_{-1}^{\mathrm{i}}\, B^*\,, \qquad \mathscr{K}_{-1}^{\mathrm{i}} := (\mathscr{L}_2^{\mathrm{i}})^{-1}\,\nabla\times \equiv \mathscr{K}_{-1}^{\mathrm{i}}\mathscr{P}\,. \tag{2.10}$$

This relation and Lemma 6 are essential because they allow to interpret all terms implying $\nabla\times B$ inside (1.16) as acting on B^* like operators of order -1 (instead of 1 when $d_e = 0$). This means that the expressions B and B^* do not play similar roles. At the same time, this invites to reconsider the hierarchy of terms when looking at (1.16). With this in mind, in the next subsection, we apply the curl operator on the first equation of (1.16).

2.3. **Transformation of the incompressible equations.** The purpose of this subsection is twofold. First, in Paragraph 2.3.1, we exploit (1.15) and (1.17) in order to recast (1.16). Secondly, in Paragraph 2.3.2, we give a concrete meaning to the notion of inertial waves.

2.3.1. *The incompressible vorticity formulation.* The point is to implement the vorticity $w := \nabla \times v$ as a new unknown. From (6.2) and (6.3), we can extract a *derived system* on $U^{\mathfrak{i}}_{\mathfrak{v}} := (w, B^*)$, which is

$$\begin{cases} \partial_t w + (v \cdot \nabla) w + (\mathscr{K}^{\mathfrak{i}}_{-1}\, B^* \cdot \nabla) B^* = \mathscr{S}^{\mathfrak{i}w}_{\mathfrak{v}0}\, U^{\mathfrak{i}}_{\mathfrak{v}} \,, \\ \partial_t B^* + ((v - d\, \mathscr{K}^{\mathfrak{i}}_{-1}\, B^*) \cdot \nabla) B^* + (\mathscr{K}^{\mathfrak{i}}_{-1}\, B^* \cdot \nabla) w = \mathscr{S}^{\mathfrak{i}B^*}_{\mathfrak{v}0}\, U^{\mathfrak{i}}_{\mathfrak{v}} \,. \end{cases} \tag{2.11}$$

In (2.11), the velocity v must be deduced from w through the Biot-Savart law (6.5), while the operator $\mathscr{S}^{\mathfrak{i}}_{\mathfrak{v}0} = (\mathscr{S}^{\mathfrak{i}w}_{\mathfrak{v}0}, \mathscr{S}^{\mathfrak{i}B^*}_{\mathfrak{v}0})$ is given by

$$\begin{aligned} &\mathscr{S}^{\mathfrak{i}w}_{\mathfrak{v}0}\, U^{\mathfrak{i}}_{\mathfrak{v}} := (B^* \cdot \nabla) \mathscr{K}^{\mathfrak{i}}_{-1}\, B^* + \sum_{i=1}^{3} w_i\, \mathcal{M}^{\mathfrak{i}}_i(w) \,, \\ &\mathscr{S}^{\mathfrak{i}B^*}_{\mathfrak{v}0}\, U^{\mathfrak{i}}_{\mathfrak{v}} := - d\, (B^* \cdot \nabla)(\mathscr{K}^{\mathfrak{i}}_{-1}\, B^*) + (w \cdot \nabla)(\mathscr{K}^{\mathfrak{i}}_{-1}\, B^*) + \sum_{i=1}^{3} B^*_i\, \mathcal{M}^{\mathfrak{i}}_i(w) \,, \end{aligned}$$

with $\mathcal{M}^{\mathfrak{i}}_i$ defined as in Lemma 23. By combining Lemmas 6 and 23, we obtain that $\mathscr{S}^{\mathfrak{i}}_{\mathfrak{v}0}$ is a (non linear) pseudo-differential operator of order zero. Hence, it can be viewed as a source term. Observe that B has disappeared from (2.11). There is no longer any need for (1.17), whereas (1.15) becomes

$$\nabla \cdot w = 0 \,, \qquad \nabla \cdot B^* = 0 \,. \tag{2.12}$$

For $0 < d_e \ll 1$, the inertial modifications appear at the level of (1.6) as perturbative terms. As such, the impact of inertial terms could seem to be marginal. But this is not so:

- At high frequencies (for $|\xi| \geq 1/d_e$), as suggested by (1.8), the inertial contributions compete with the other influences.
- The constitutive relation (1.5) induces (through Lemma 6) a complete reordering of the unknowns. The change is brutal from $d_e = 0$ to $d_e > 0$. Once $d_e > 0$, the terms which manage in standard MHD the Alfven and Magnetosonic waves are relegated inside the source term $\mathscr{S}^{\mathfrak{i}}_{\mathfrak{v}0}\, U^{\mathfrak{i}}_{\mathfrak{v}}$, where they play the role of zero order contributions. Still, they participate to lower order dispersive effects.
- In XMHD, new terms become predominant. Emphasis is given to the symmetric part which, in the left part of (2.11), involves $\mathscr{K}^{\mathfrak{i}}_{-1}$.

In other words, the passage from (1.6) to (1.8), and especially from (1.8) to (2.11), is very singular (there is no smooth transition from $d_e = 0$ to $d_e > 0$). It makes appear the (hidden) hyperbolic structure of XMHD. The consequence in terms of the occurrence and organization of waves is as explained just after Theorem 2.

Remark 7. *[Energy spectra] In* [3, 38]*, using a Kolmogorov-like analysis and hypotheses (regarding the energy and helicity cascades), the authors obtain the energy spectra of XMHD in different (ideal, Hall and inertial) regimes. This study confirms that many types of waves overlap in XMHD, while inertial features can overtake at high frequencies.*

2.3.2. *Inertial waves.* Excluding for the moment the coupling induced by the source terms and assuming that $d_i = 0$ (so that $d = 0$), the system (2.11) reduces to

$$\left\{ \begin{array}{l} \partial_t w + (v \cdot \nabla) w + (\mathscr{K}^{\mathrm{i}}_{-1} B^* \cdot \nabla) B^* = 0 \,, \\ \partial_t B^* + (v \cdot \nabla) B^* + (\mathscr{K}^{\mathrm{i}}_{-1} B^* \cdot \nabla) w = 0 \,. \end{array} \right. \tag{2.13}$$

Noting $B^0_\pm := B^* \pm w$, this is the same as a nonlinear coupled system of two transport equations, namely

$$\partial_t B^0_\pm + (v \cdot \nabla) B^0_\pm \pm \frac{1}{2} \, (\mathscr{K}^{\mathrm{i}}_{-1} (B^0_+ + B^0_-) \cdot \nabla) B^0_\pm = 0 \,.$$

We can immediately recognize two distinct eigenvalues (which provide a first access to inertial waves), each of multiplicity 3, which are

$$\lambda_\pm \equiv \lambda_\pm(v, B^0_+, B^0_-, \xi) := v \cdot \xi \pm \frac{1}{2} \, \mathscr{K}^{\mathrm{i}}_{-1} (B^0_+ + B^0_-) \cdot \xi \,. \tag{2.14}$$

These eigenvalues $\lambda_\pm$ are formally genuinely nonlinear in the sense that

$$(\tilde{B}^0_\pm \cdot \nabla_{B^0_\pm}) \lambda_\pm(v, B^0_+, B^0_-, \xi) = \pm \frac{1}{2} \, \mathscr{K}^{\mathrm{i}}_{-1} \, \tilde{B}^0_\pm \, \cdot \xi \not\equiv 0 \,.$$

It turns out that the above elementary diagonalisation procedure can be generalized to the whole system. Indeed, with

$$B^d_\pm := B^* + \kappa^d_\pm \, \nabla \times v \,, \qquad v^d_\pm := v - \kappa^d_\mp \, \nabla \times B \,, \qquad \kappa^d_\pm := \frac{1}{2} \, \Big(d \pm \sqrt{d^2 + 4} \Big) \,, \tag{2.15}$$

the incompressible XMHD equations (1.16) can be recast as

$$\partial_t B^d_\pm + \nabla \times (B^d_\pm \times v^d_\pm) = 0 \,. \tag{2.16}$$

The formulation (2.15)-(2.16) is implicit in [3, 31]. From (2.15), following the preceding lines, we can extract

$$v^d_\pm = \nabla \times (-\Delta)^{-1} \Big(\frac{B^d_+ - B^d_-}{\kappa^d_+ - \kappa^d_-} \Big) - \kappa^d_\mp \, \nabla \times (1 - \Delta)^{-1} \Big(\frac{\kappa^d_+ \, B^d_- - \kappa^d_- \, B^d_+}{\kappa^d_+ - \kappa^d_-} \Big) \,. \tag{2.17}$$

In other words, incompressible XMHD can also be seen as two incompressible transport equations on $B^d_\pm$ with velocities $v^d_\pm$, where the latter are given in terms of $B^d_\pm$ by the generalized Biot–Savart type laws (2.17). The unknowns $B^d_\pm = B^* + \kappa^d_\pm \, w$ are made of adequate linear combinations of B^* and w, together with a link to v and therefore $v_\pm$ (in order to close the system). As in (2.11), the unknowns are in fact the components of $U^{\mathrm{i}}_{\mathfrak{v}}$. As in (2.11), the system (2.16) completed with (2.17) is a quasilinear symmetric system whose both coefficients and source terms take the form of zero order pseudo-differential operators. Working with (2.13) or (2.16) are two equivalent options. In this text, we select the approach through (2.13).

Select some special solution $(\bar{w}, \bar{B}^*)$ to the system (2.13). We can consider the (one order part of the) linearized equations along $(\bar{w}, \bar{B}^*)$ associated with (2.11), which are

$$\left\{ \begin{aligned} & \partial_t \dot{w} + (\bar{v}\cdot\nabla)\dot{w} + (\mathscr{K}^{\mathfrak{i}}_{-1}\,\bar{B}^*\cdot\nabla)\dot{B}^* = 0\,, \\ & \partial_t \dot{B}^* + \big((\bar{v} - d\,\mathscr{K}^{\mathfrak{i}}_{-1}\,\bar{B}^*)\cdot\nabla\big)\dot{B}^* + \big(\mathscr{K}^{\mathfrak{i}}_{-1}\,\bar{B}^*\cdot\nabla\big)\dot{w} = 0\,. \end{aligned} \right. \tag{2.18}$$

Definition 8. *The inertial waves (related to the choice of $\bar{w}$ and $\bar{B}^*$) are carried by the two eigenvalues $\lambda_\pm(\xi)$ which are each with multiplicity 3 of the linear hyperbolic system (2.18), namely*

$$\lambda_\pm \equiv \lambda_\pm(\xi) := v\cdot\xi - \kappa^d_\pm\;(\mathscr{K}^{\mathfrak{i}}_{-1}\,\bar{B}^*)\cdot\xi\,, \qquad \kappa^d_\pm := \frac{1}{2}\;\big(d \pm \sqrt{d^2+4}\big)\,. \tag{2.19}$$

To observe experimentally inertial waves, two conditions must be fulfilled:

- The plasma must be sufficiently energetic to trigger high frequencies $|\xi| \geq 1/d_e$.
- The data must be expressed in terms of w and B^* (or even better $B^d_\pm$). Indeed, information collected just in terms of v could be difficult to interpret.

2.4. **Proof of Theorem 1.** We start by showing Theorem 1 under the more restrictive regularity assumption $s > 7/2$. We refer to Section 4 for the optimal result. In particular, at time $t = 0$, we know that (with $\tilde{s} := s - 1$)

$$U^{\mathfrak{i}}_{\mathfrak{v}}(0,\cdot) = U^{\mathfrak{i}}_{\mathfrak{v}0} = (w_0, B^*_0) \in \mathcal{D}^{\tilde{s}}(\mathbb{R}^3;\mathbb{R}^3)^3\,, \quad w_0 := \nabla\times v_0\,, \quad \tilde{s} > 5/2\,. \tag{2.20}$$

Any smooth solution to (1.15)-(1.16)-(1.17)-(1.18) leads to a solution to (2.11)-(2.12)-(2.20), and conversely. We study below the time evolution of the L^2-norm of $U^{\mathfrak{i}}_{\mathfrak{v}}$ (assuming for the moment that $U^{\mathfrak{i}}_{\mathfrak{v}}$ is bounded in the large $\mathcal{H}^{\tilde{s}}$-norm).

Lemma 9. *[L^2-energy estimate for the incompressible vorticity formulation] Let $T > 0$. Assume that the function $U^{\mathfrak{i}}_{\mathfrak{v}} \in C([0,T];\mathcal{D}^{\tilde{s}})$ with $\tilde{s} > 5/2$ is a solution to (2.11) with initial data (2.20). Then, we can find a constant C depending only on the $C([0,T];\mathcal{H}^{\tilde{s}})$-norm of $U^{\mathfrak{i}}_{\mathfrak{v}}$ such that*

$$\parallel U^{\mathfrak{i}}_{\mathfrak{v}}(t,\cdot)\parallel_{L^2} \leq \parallel U^{\mathfrak{i}}_{\mathfrak{v}0}\parallel_{L^2}\; e^{C\,t}\,, \qquad \forall\, t\in[0,T]\,. \tag{2.21}$$

Proof. Multiply the first and second equation of (2.11) respectively by w and B^*, and then integrate with respect to x. Since $v \in \mathcal{D}^{\tilde{s}}$, the contributions issued from the (transport) diagonal part involving $v\cdot\nabla$ disappear. After integrations by parts, there remains

$$\begin{aligned} \frac{1}{2}\;\frac{d}{dt}\Big(\int_{\mathbb{R}^3} |U^{\mathfrak{i}}_{\mathfrak{v}}(t,\cdot)|^2\;dx\Big) = &-\frac{d}{2}\int_{\mathbb{R}^3} \nabla\cdot\big(\mathscr{K}^{\mathfrak{i}}_{-1}\,B^*\big)\;|B^*|^2\;dx \\ &+ \int_{\mathbb{R}^3} \nabla\cdot\big(\mathscr{K}^{\mathfrak{i}}_{-1}\,B^*\big)\;(w\cdot B^*)\;dx + \int_{\mathbb{R}^3} U^{\mathfrak{i}}_{\mathfrak{v}}\cdot\mathscr{S}^{\mathfrak{i}}_{\mathfrak{v}0} U^{\mathfrak{i}}_{\mathfrak{v}}\,dx\,. \end{aligned}$$

The Fourier multiplier $(\mathscr{L}^{\mathfrak{i}}_2)^{-1}$ commutes with $\nabla\cdot$, while $\nabla\cdot\nabla\times \equiv 0$. Thus

$$\frac{1}{2}\;\frac{d}{dt}\Big(\int_{\mathbb{R}^3} |U^{\mathfrak{i}}_{\mathfrak{v}}(t,\cdot)|^2\,dx\Big) = \int_{\mathbb{R}^3} U^{\mathfrak{i}}_{\mathfrak{v}}\cdot\mathscr{S}^{\mathfrak{i}}_{\mathfrak{v}0} U^{\mathfrak{i}}_{\mathfrak{v}}\;dx\,.$$

Below, we use the Sobolev embedding theorem $\mathcal{H}^{\tilde{s}} \hookrightarrow L^\infty$ (knowing that $\tilde{s} > 5/2$). We exploit the condition $\nabla \cdot v = 0$ to deal with the sum of products $w_i \ \mathcal{M}_i^{\mathfrak{i}}(w)$. We also implement Lemmas 6 and 23 to get

$$\begin{aligned}\Big| \int_{\mathbb{R}^3} w \cdot \mathscr{S}_{\mathfrak{v}0}^{\mathfrak{i}w} U_{\mathfrak{v}}^{\mathfrak{i}} \, dx \Big| &\leq \parallel w \parallel_{L^\infty} \ \sum_{i=1}^3 (\parallel B_i^* \parallel_{L^2} \parallel \partial_i \mathscr{K}_{-1}^{\mathfrak{i}} B^* \parallel_{L^2} + \parallel w_i \parallel_{L^2} \parallel \mathcal{M}_i^{\mathfrak{i}}(w) \parallel_{L^2}) \\ &\lesssim \parallel U_{\mathfrak{v}}^{\mathfrak{i}} \parallel_{C([0,T];\mathcal{H}^{\tilde{s}})} \parallel U_{\mathfrak{v}}^{\mathfrak{i}} \parallel_{L^2}^2 ,\end{aligned}$$

as well as

$$\begin{aligned}\Big| \int_{\mathbb{R}^3} B^* \cdot \mathscr{S}_{\mathfrak{v}0}^{\mathfrak{i}B^*} U_{\mathfrak{v}}^{\mathfrak{i}} \, dx \Big| &\leq d \ \parallel B^* \parallel_{L^\infty} \sum_{i=1}^3 \parallel B_i^* \parallel_{L^2} \parallel \partial_i \mathscr{K}_{-1}^{\mathfrak{i}} B^* \parallel_{L^2} \\ &\quad + \parallel B^* \parallel_{L^\infty} \sum_{i=1}^3 (\parallel w_i \parallel_{L^2} \parallel \partial_i \mathscr{K}_{-1}^{\mathfrak{i}} B^* \parallel_{L^2} + \parallel B_i^* \parallel_{L^2} \parallel \mathcal{M}_i^{\mathfrak{i}}(w) \parallel_{L^2}) \\ &\lesssim \parallel U_{\mathfrak{v}}^{\mathfrak{i}} \parallel_{C([0,T];\mathcal{H}^{\tilde{s}})} \parallel U_{\mathfrak{v}}^{\mathfrak{i}} \parallel_{L^2}^2 \ .\end{aligned}$$

By Grönwall's inequality, we recover (2.21). □

The proof of Lemma 9 serves to confirm that the source term is indeed of order 0. To go further, we have to write down a scheme [6, 33] in order to use a fixed-point method. To this end, we need to implement the linearized version of (2.11). Then, we have to perform energy estimates in order to obtain a control in the large norm $L^\infty([0,T];\mathcal{H}^{\tilde{s}})$, and a convergence in the small norm $L^\infty([0,T];L^2)$.
When doing this, the coefficients (which are transparent in the above proof) are implied. The only difficulty could come from the operator $(\mathscr{K}_{-1}^{\mathfrak{i}} B^*) \cdot \nabla$ but the coefficient $\mathscr{K}_{-1}^{\mathfrak{i}} B^*$ is of order 0 (and even of order -1) as required. Thus, L^2-energy estimates are available for the linearized equations along the same lines as above.
To get $\mathcal{H}^{\tilde{s}}$-bounds, we have to commute the linearized equation with spatial derivatives ∂_x^α with $|\alpha| \leq \tilde{s}$, and exploit linear estimates of nonlinear functions. This falls under the scope of the general strategy [6, 33] to solve quasilinear symmetric systems. The details, which are standard and long, are not reproduced here. The conclusion is that the Cauchy problem associated with (2.11) is well-posed in $\mathcal{H}^{\tilde{s}}$ for $\tilde{s} > 5/2$.
From the $\mathcal{H}^{\tilde{s}}$-solutions to (2.11) with $\tilde{s} > 5/2$, we recover solutions to (1.16), which are such that $(v, B^*)(t,\cdot) \in \mathcal{H}^s \times \mathcal{H}^{s-1}$ with $s := \tilde{s} + 1 > 7/2$. Moreover, from Lemma 6 together with (2.10), we obtain that $B(t,\cdot) \in \mathcal{H}^{s+1}$. This concludes the proof of Theorem 1 at least on condition that $s > 7/2$.

Remark 10. *[Propagated L^2-energy for (2.11) versus conserved quantity for (1.16)] From Lemma 6, we know that $\parallel \nabla \times B \parallel_{L^2} = \parallel \mathscr{K}_{-1}^{\mathfrak{i}} B^* \parallel_{L^2} \lesssim \parallel B^* \parallel_{L^2}$. It is clear that, with $\mathcal{E}^{\mathfrak{i}}$ as in (2.1), we have $\mathcal{E}^{\mathfrak{i}} \lesssim \parallel U_{\mathfrak{v}}^{\mathfrak{i}} \parallel_{L^2}$. The opposite is false. In other words, Lemma 9 is not a corollary of Lemma 3.*

3. The compressible framework

In this section, $p : \mathbb{R}_+ \to \mathbb{R}$ is a given strictly increasing smooth function of ρ. We extend here (1.18) by putting aside the condition $\nabla \cdot B_0^* = 0$. As a matter of fact, we consider general vector fields B^*. This is made possible by the following remark.

Lemma 11. *[Conservation of the magnetic divergence] Any solution to (1.8)-(1.9)-(1.10) is such that*

$$\nabla \cdot B^* = \nabla \cdot B = \nabla \cdot B_0^* . \tag{3.1}$$

Proof. This is just because $\partial_t(\nabla \cdot B^*) = 0$. □

The whole vector field B^* (resp. B) can be reconstituted from $\nabla \cdot B^*$ and $\nabla \times B^*$ (resp. from $\nabla \cdot B$ and $\nabla \times B$) by solving the div-curl system (see Subsection 6.2). The parts $\nabla \cdot B^*$ and $\nabla \cdot B$ are determined by (3.1). In particular, with $\mathscr{P} = P(D_x)$ where P is as in (2.7), retain that

$$B^* = \mathscr{Q}\, B_0^* + \mathscr{P}\, B^* , \qquad \mathscr{Q} = \mathrm{Id} - \mathscr{P} . \tag{3.2}$$

In other words, replacing everywhere B^* as indicated above, the system (1.8) reduces to an equation on $(\rho, v, \mathscr{P} B^*)$, while the constitutive relation (1.9) is aimed to deduce $\mathscr{P} B$ from $\mathscr{P} B^*$, or equivalently $\nabla \times B$ from $\nabla \times B^*$.

Lemma 11 is straightforward. It is however highlighted because it plays a crucial role for the reason explained in the remark below.

Remark 12. *[A consequence of the foliation by vector fields having a fixed divergence] Let C be a smooth vector field viewed as a coefficient. From (6.2), we have the decomposition*

$$\mathscr{T}_C\, B^* \equiv \mathscr{T}\, B^* := \nabla \times (C \times B^*) = \mathscr{T}_1\, B^* + \mathscr{T}_0\, B^* \tag{3.3}$$

with

$$\mathscr{T}_1\, B^* := (\nabla \cdot B^*)\ C - (C \cdot \nabla) B^* , \qquad \mathscr{T}_0\, B^* := (B^* \cdot \nabla) C - (\nabla \cdot C)\ B^* . \tag{3.4}$$

The operator $\mathscr{T}_0$ is of order 0, while $\mathscr{T}_1$ is of order 1. The action of $\mathscr{T}_1$ is not skew-adjoint. As such, it is not compatible with energy estimates. However, knowing (3.1), we should opt for $\mathscr{T}\, B^* = \tilde{\mathscr{T}}_1\, B^* + \tilde{\mathscr{T}}_0\, B^*$ with

$$\tilde{\mathscr{T}}_1\, B^* := -(C \cdot \nabla) B^* , \qquad \tilde{\mathscr{T}}_0\, B^* := (\nabla \cdot B_0^*)\ C + (B^* \cdot \nabla) C - (\nabla \cdot C)\ B^* . \tag{3.5}$$

The operator $\tilde{\mathscr{T}}_1$ is skew-adjoint. Contrary to $\mathscr{T}_1$, it can be dealt with in the energy estimates without losses of derivatives. This trick will be repeatedly used. As a matter of fact, we will systematically replace $\nabla \cdot B^*$ by $\nabla \cdot B_0^*$.

In order to make the transition from $\nabla \times B^*$ to $\nabla \times B$, we have to exploit conveniently the constitutive relation (1.9). To this end, we follow a plan similar to Section 2. In Subsection 3.1, we come back to the content of (1.9) but this time when ρ is a non constant function. In Subsection 3.2, we adapt to the compressible context the change of variables of Subsection 2.3. In Subsection 3.3, we derive energy estimates to show Theorem 2 (for $s > 7/2$). At each stage, in comparison with Section 2, we need to implement important and difficult modifications.

3.1. **The compressible constitutive relation.** The difficulty here is to exploit (1.9) in order to express $\mathscr{P}B$ in terms of $\mathscr{P}B^*$. In this subsection, we fix a time $t \in \mathbb{R}_+$, and we assume that the function $\rho(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}$ is bounded and positive. More precisely, we impose

$$\exists\,(c,C) \in \mathbb{R}^2\,; \qquad 0 < c \leq \rho(t,x) \leq C\,, \qquad \forall\, x \in \mathbb{R}^3\,. \tag{3.6}$$

We also suppose that the function $\rho(t,\cdot)$ is smooth enough, say in $\mathcal{H}^s(\mathbb{R}^3)$ with $s > 7/2$. By this way, we can use the pseudo-differential calculus with coefficients in $\mathcal{H}^s$, as developed for instance in [34, 42]. In what follows, we will sometimes omit to mention the presence of t. From (1.9), we get that

$$\nabla \times B^* = \mathscr{L}_2^{\mathfrak{c}} \Big(\frac{\nabla \times B}{\rho(x)} \Big)\,, \qquad \mathscr{L}_2^{\mathfrak{c}} := \rho(x)\,\mathrm{Id} + \nabla \times \nabla \times\ . \tag{3.7}$$

We look at $\mathscr{L}_2^{\mathfrak{c}} : L^2(\mathbb{R}^3;\mathbb{R}^3) \to L^2(\mathbb{R}^3;\mathbb{R}^3)$ as an unbounded operator [12].

Lemma 13. *[Inverse of $\mathscr{L}_2^{\mathfrak{c}}$] The operator $(\mathscr{L}_2^{\mathfrak{c}})^{-1} : L^2 \to L^2$ is well-defined and bounded.*

Proof. Observe that $\mathscr{L}_2^{\mathfrak{c}}$ is symmetric and positive since

$$\begin{aligned} \int (\mathscr{L}_2^{\mathfrak{c}} u)(x) \cdot \bar{u}(x)\ dx = & \int \rho(x)\ |u(x)|^2\ dx \\ & + \int |\nabla \times u(x)|^2\ dx \geq c\ \parallel u \parallel_{L^2}^2\,, \qquad \forall u \in \mathcal{H}^2\,. \end{aligned} \tag{3.8}$$

The polar decomposition furnishes the existence of a densely defined, closed and self-adjoint operator $\mathscr{X}^{\mathfrak{c}} : L^2 \to L^2$ with domain $\mathrm{Dom}\,(\mathscr{X}^{\mathfrak{c}})$ such that $\mathscr{L}_2^{\mathfrak{c}} = (\mathscr{X}^{\mathfrak{c}})^* \,\mathscr{X}^{\mathfrak{c}}$, and therefore

$$\begin{aligned} &\parallel \mathscr{X}^{\mathfrak{c}}\, u \parallel_{L^2} \geq \sqrt{c}\ \parallel u \parallel_{L^2}\,, \qquad \forall\, u \in \mathrm{Dom}\,(\mathscr{X}^{\mathfrak{c}})\,, \\ &\parallel (\mathscr{X}^{\mathfrak{c}})^*\, u \parallel_{L^2} \geq \sqrt{c}\ \parallel u \parallel_{L^2}\,, \qquad \forall\, u \in \mathrm{Dom}\,\big((\mathscr{X}^{\mathfrak{c}})^*\big)\,. \end{aligned}$$

Starting from there, the two operators $\mathscr{X}^{\mathfrak{c}}$ and $(\mathscr{X}^{\mathfrak{c}})^*$ are invertible (Theorem 3.3.2 in [15], or see also [12]). The same applies to $\mathscr{L}_2^{\mathfrak{c}}$ with $(\mathscr{L}_2^{\mathfrak{c}})^{-1} = (\mathscr{X}^{\mathfrak{c}})^{-1} \circ \big((\mathscr{X}^{\mathfrak{c}})^*\big)^{-1}$. □

For smooth enough vector fields B^*, the relation (3.7) amounts to the same thing as

$$\frac{\nabla \times B}{\rho(x)} = \mathscr{K}_{-1}^{\mathfrak{c}}\, B^*\,, \qquad \mathscr{K}_{-1}^{\mathfrak{c}} := (\mathscr{L}_2^{\mathfrak{c}})^{-1}\, \nabla \times \equiv \mathscr{K}_{-1}^{\mathfrak{c}} \mathscr{P}\,. \tag{3.9}$$

The system (1.8) where $\nabla \times B/\rho$ is replaced everywhere as indicated in (3.9) is enough to recover a self-contained system on ${}^t(\rho, v, \mathscr{P}B^*)$. We can progress without introducing B and without imposing $\nabla \cdot B^* = 0$. Neither (1.9) nor (1.15) are needed. It suffices to rely on (3.9). Still, the passage through (1.9) and (1.15), which is prescribed by physicists, is meaningful. First, it is a way to deduce the final constitutive relation (3.9). Secondly, it is more adapted in view of the potential formulation (in Section 4). Now, one important key in continuity with Lemma 6 is to show that the restriction of $(\mathscr{L}_2^{\mathfrak{c}})^{-1}$ to $\mathcal{D}^r$ (for well-chosen indices r) still gives rise to a gain of two derivatives. In other words, we have to justify the subscript -1 in $\mathscr{K}_{-1}^{\mathfrak{c}}$.

Proposition 14 (A property of ellipticity when going from $\nabla \times B^*$ to $\nabla \times B/\rho$ through the constitutive relation (3.9))**.** *The action of* $(\mathscr{L}_2^{\mathfrak{c}})^{-1}$ *is associated with a matrix valued operator whose all coefficients are pseudo-differential operators. Its restriction to solenoidal vector fields is elliptic of order less or equal to* -2*. More precisely, for* $r \in [s-2, s]$*, the action* $(\mathscr{L}_2^{\mathfrak{c}})^{-1} : \mathcal{D}^r \to \mathcal{H}^{r+2}$ *is well-defined and continuous.*

In comparison with Lemma 6, the variations of the function ρ induce modifications:

- *d1.* First, unlike $(\mathscr{L}_2^{\mathfrak{i}})^{-1}$, the image of $(\mathscr{L}_2^{\mathfrak{c}})^{-1}$ on $\mathcal{D}^r$ is not $\mathcal{D}^{r+2}$. Indeed, since ρ is not constant, the action of $(\mathscr{L}_2^{\mathfrak{c}})^{-1}$ implies a deformation out of the set of solenoidal vector fields. This is restored as indicated in (3.9) after multiplication by ρ. The transition from $\nabla \times B^*$ to $\nabla \times B$ through (3.9) is not diagonal; it is not so simple. In particular, the identity

$$\nabla \times B = \rho(x) \; (\tilde{\mathscr{L}}_2^{\mathfrak{c}})^{-1}(\nabla \times B^*) \,, \qquad \tilde{\mathscr{L}}_2^{\mathfrak{c}} := \big(\rho(x) - \Delta\big) \; \mathrm{Id}_{3\times 3} \,, \tag{3.10}$$

  which could appear as the correct extrapolation of (1.17) is false.
- *d2.* Secondly, there are restrictions on r which are absent (on s) at the level of (2.4). On the one hand, the upper bound $r \leq s$ comes from the limited regularity of ρ. On the other hand, the lower bound $s-2 \leq r$ is issued from the rules of composition in Sobolev spaces (in view of a para-differential calculus). We can further illustrate these two conditions by looking at the elliptic equation $\big(\rho(x) - \Delta\big)\, u = f$ where $u \in \mathcal{H}^{s-1}$ and $f \in \mathcal{H}^r$ with $r \in [s-2, s]$. Then, knowing that $s > 7/2$, from Theorem 1.2.A in [42], we obtain that $u \in \mathcal{H}^{r+2}$.

Proof. The rest of Subsection 3.1 is devoted to the proof of Proposition 14. To overcome the difficulty *d1*, we must keep track of derivative losses concerning ρ. To solve *d2*, we follow a procedure in three steps:

- In Paragraph 3.1.1, we explain our method of unitary conjugation, and we introduce preliminary tools like the Weyl quantization.
- In Paragraph 3.1.2, we show by Weyl calculus that $\mathscr{L}_2^{\mathfrak{c}}$ is (almost) unitary equivalent to a block diagonal action. In fact, the principal symbol of $\mathscr{L}_2^{\mathfrak{c}}$ has two distinct eigenvalues: $\rho(x)$ and $\rho(x) + |\xi|^2$ which are respectively of multiplicity one and two. The unitary reduction reveals a 2×2 elliptic block of order 2, corresponding to the second eigenvalue and involving (after inversion) a gain of two derivatives. The difficulty is to show that this gain remains effective on $\mathcal{D}^r$, while it could be destroyed by the variations of ρ. The presence of a non constant function ρ produces nonzero commutators and by this way non diagonal terms. In this line, note again that the relation (3.10) is not verified.
- In Paragraph 3.1.3, to remedy this, we construct an approximate parametrix, and we check that its properties allow to conclude.

□

3.1.1. *Preparatory work.* We denote by $OP\mathcal{H}^sS^m$ the set of pseudo-differential operators of order less or equal to m with symbols in $\mathcal{H}^s$ (e.g., see [34, 42]), and simply $\mathrm{Op}(m)$ an element of $OP\mathcal{H}^rS^m$ (for some unspecified $r = s-1$ or $r = s$). In view of (3.8), the action of $\mathscr{L}_2^{\mathfrak{c}}$ is (at least) elliptic of order 0. Thus, to evaluate its precise order, it suffices to consider what happens for large frequencies, that is for ξ with $|\xi| \gg 1$. The action of $\mathscr{L}_2^{\mathfrak{c}}$ is achieved through a matrix valued differential operator, which is non diagonal. In line with (2.5) and (2.6), a first attempt to obtain a block diagonal form is to look at

$$\mathscr{O}_0^{-1}\,\mathscr{L}_2^{\mathfrak{c}}\,\mathscr{O}_0 = \mathscr{D}^{\mathfrak{c}} + \mathscr{E}\,, \qquad \mathscr{E} := \mathscr{O}_0^{-1}\left[\rho(x)\,\mathrm{Id}, \mathscr{O}_0\right],$$

where $\mathscr{D}^{\mathfrak{c}}$ and $\mathscr{E} = \mathscr{E}^*$ are given by

$$\mathscr{D}^{\mathfrak{c}} := \begin{pmatrix} \rho(x) & 0 & 0 \\ 0 & \rho(x)-\Delta & 0 \\ 0 & 0 & \rho(x)-\Delta \end{pmatrix}, \qquad \mathscr{E} = \begin{pmatrix} \mathscr{E}_{11} & \mathscr{E}_{12} & \mathscr{E}_{13} \\ \mathscr{E}_{12}^* & \mathscr{E}_{22} & \mathscr{E}_{23} \\ \mathscr{E}_{13}^* & \mathscr{E}_{23}^* & \mathscr{E}_{33} \end{pmatrix}.$$

It is clear that $\mathscr{D}^{\mathfrak{c}} \in OP\mathcal{H}^s S^2$. Thus, in the absence of $\mathscr{E}$, Proposition 14 would be a direct consequence of Theorem 1.2.A in [42]. There remains to explain how to absorb the above remainder $\mathscr{E}$.

Denoting by $\mathscr{O}_{0ij} = O_{0ij}(D_x)$ the elements of the matrix valued pseudo-differential operator $\mathscr{O}_0$ (which are all of order 0), we find that (e.g., see Corollary 4.1 in [6])

$$(\mathscr{E})_{ij} = \mathscr{O}_0^{-1}\Big(\left[\rho(x), \mathscr{O}_{0ij}\right]\Big)_{ij} = \mathscr{O}_0^{-1}\Big(\mathrm{Op}(i\,\{O_{0ij}(\xi), \rho(x)\})\Big)_{ij} + \mathrm{Op}(-2)\,,$$

where we have introduced the Poisson bracket, which is given in the phase space (x,ξ) by

$$\{f,g\} := \sum_{i=1}^{3} \partial_{\xi_i} f\ \partial_{x_i} g - \sum_{i=1}^{3} \partial_{x_i} f\ \partial_{\xi_i} g\,.$$

We see on this formula that $\mathscr{E} \in OP\mathcal{H}^{s-1}S^{-1}$. The above reduction is not yet sufficient in order to conclude (due to the presence of non zero coefficients $\mathscr{E}_{1\star}$). To (partially) further absorb $\mathscr{E}$, the idea is to find a unitary operator $\mathscr{V}$ such that

$$\mathscr{V}^*\left(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}\right)\mathscr{V} = \mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r + \mathrm{Op}(-4)\,, \quad \mathscr{E}_r := \begin{pmatrix} \mathscr{E}_{11} & 0 & 0 \\ 0 & \mathscr{E}_{22} & \mathscr{E}_{23} \\ 0 & \mathscr{E}_{23}^* & \mathscr{E}_{33} \end{pmatrix} = \mathscr{E}_r^*\,. \tag{3.11}$$

To this end, we seek $\mathscr{V}$ in the form $\mathscr{V} = e^{i\mathscr{A}}$ where $\mathscr{A}$ is a self-adjoint pseudo-differential operator with real valued symbol A. When doing this, to facilitate calculations, it is more appropriate to work with the Weyl quantization (with symbol A) given by

$$\mathscr{A}\,u(x) \equiv \mathrm{Op}^W(A)u(x) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(x-y)\cdot\xi}\ A\Big(\frac{x+y}{2},\xi\Big)\ u(y)\ dy\,d\xi\,, \quad u \in \mathcal{D}(\mathbb{R}^3)\,.$$

We recall that any operator $\mathscr{A}$ having a real symbol A is self-adjoint, so that $e^{i\mathscr{A}}$ is a unitary pseudo-differential operator satisfying

$$e^{i\mathscr{A}} = \sum_{k=0}^{+\infty} \frac{i^k}{k!}\,\mathscr{A}^k = \mathrm{Id} + i\,\mathscr{A} + \cdots\,, \qquad (e^{i\mathscr{A}})^* = e^{-i\mathscr{A}}\,.$$

When $\mathscr{A}$ is of negative order ($m<0$), the above sum implements terms $\mathscr{A}^k$ which are of decreasing orders $k\,m$. For instance, the above remainder (marked by $\cdots$) is in $\mathrm{Op}(2\,m)$.

3.1.2. *Unitary reduction.* Assume that $\mathscr{A} = (\mathscr{A}_{ij})_{ij}$ is a self-adjoint operator of negative order -3. Then, we deal with

$$\begin{aligned} \mathscr{V}^* \left(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}\right) \mathscr{V} &= \big(\mathrm{Id} - i\mathscr{A} + \mathrm{Op}(-6)\big) \left(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}\right) \big(\mathrm{Id} + i\mathscr{A} + \mathrm{Op}(-6)\big) \\ &= \mathscr{D}^{\mathfrak{c}} + \mathscr{E} + i \left[\mathscr{D}^{\mathfrak{c}}, \mathscr{A}\right] + i \left[\mathscr{E}, \mathscr{A}\right] + \mathscr{A} \left(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}\right) \mathscr{A} + \mathrm{Op}(-4) \\ &= \mathscr{D}^{\mathfrak{c}} + \mathscr{E} + i \left[\mathscr{D}^{\mathfrak{c}}, \mathscr{A}\right] + \mathrm{Op}(-4) \,. \end{aligned}$$

Thus, to recover (3.11), we have to consider the homological equation

$$i \left[\mathscr{D}^{\mathfrak{c}}, \mathscr{A}\right] = \begin{pmatrix} 0 & -\mathscr{E}_{12} & -\mathscr{E}_{13} \\ -\mathscr{E}_{12}^* & 0 & 0 \\ -\mathscr{E}_{13}^* & 0 & 0 \end{pmatrix} + \mathrm{Op}(-4) \,,$$

where the $\mathscr{E}_{1j}$ with $j \in \{2,3\}$ are given and $\mathscr{A}$ is the unknown. We impose $\mathscr{A}_{ij} = 0$ for (i,j) not equal to $(1,2)$ or $(1,3)$. Then, we have to solve $i\ \left[\rho(x), \mathscr{A}_{1j}\right] + i\ \mathscr{A}_{1j}\ \Delta = -\mathscr{E}_{1j} + \mathrm{Op}(-4)$.where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ Assuming that $\mathscr{A}$ is in $\mathrm{Op}(-3)$, the commutator $[\rho, \mathscr{A}_{1j}]$ is in $\mathrm{Op}(-4)$, and this reduces to

$$\mathscr{A}_{1j} = i\ \mathscr{E}_{1j}\ \Delta^{-1} + \mathrm{Op}(-6) \,.$$

We just take $\mathscr{A}_{1j} := i\ \mathscr{E}_{1j}\ \Delta^{-1}$. With this choice, as required initially, the operator $\mathscr{A}$ is indeed in $OP\mathcal{H}^{s-1}S^{-3}$. Moreover, by construction, we have access to (3.11).

Retain that $\mathscr{A}$ is of small order for two reasons. First, $\mathscr{E}$ is obtained by commuting $\mathscr{O}_0$ with the diagonal matrix $\rho(x)\,\mathrm{Id}$ with a corresponding gain of one derivative, so that $\mathscr{E} \equiv \mathscr{E}_{-1}$. Secondly, the difference $|\xi|^2$ between the two eigenvalues $\rho(x)$ and $\rho(x) + |\xi|^2$ is of order two. After division, this yields a supplementary gain of two derivatives. Note also that the above process does not allow to go further in the diagonalization process, in order to get rid of $\mathscr{E}_{23}$. Indeed, the eigenvalue $\rho(x) + |\xi|^2$ is of multiplicity 2. Thus, we cannot exploit any gap between the (same two) eigenvalues related to the bottom 2×2 block.

3.1.3. *The approximate parametrix.* Look at $\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r$ where the orders are clearly separated:

- *Top* 1×1 *block.* The scalar pseudo-differential operator $\rho(x) + \mathscr{E}_{11}$ is self-adjoint. Moreover, it is elliptic of order 0 since its principal symbol is the function $\rho(x)$, which satisfies (3.6). Thus, it can be inverted, and its inverse is a self-adjoint pseudo-differential operator of order 0.
- *Bottom* 2×2 *block.* This is

$$(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{Bot}_{2\times 2} := \big(\rho(x) - \Delta\big)\ \mathrm{Id}_{2\times 2} + \mathscr{E}^{Bot}_{2\times 2} \,, \qquad \mathscr{E}^{Bot}_{2\times 2} := \begin{pmatrix} \mathscr{E}_{22} & \mathscr{E}_{23} \\ \mathscr{E}_{23}^* & \mathscr{E}_{33} \end{pmatrix} ,$$

  where by construction $\mathscr{E}^{Bot}_{2\times 2} \in OP\mathcal{H}^{s-1}S^{-1}$ acts continously on $\mathcal{H}^r$. The above operator is self-adjoint. Once $\rho(x)$ satisfies (3.6), it is extracted from an operator which is elliptic of order 0. As such, it is an elliptic operator of order 0. For large frequencies, it is (in view of its principal symbol $|\xi|^2$) elliptic of order 2. Consider (for $s-2 \le r \le s$) the elliptic equation $(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{Bot}_{2\times 2}\, u = f \in \mathcal{H}^r$, or alternatively

$$\big(\rho(x) - \Delta\big)\, u = f - \mathscr{E}^{Bot}_{2\times 2}\, u \in \mathcal{H}^r \,.$$

By applying Theorem 1.2.A in [42], we recover that $u \in \mathcal{H}^{r+2}$ as required. In conclusion, the operator $(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{Bot}_{2\times 2}$ is elliptic of order 2 on the whole phase space. It can therefore be inverted, and its inverse is a self-adjoint matrix valued pseudo-differential operator of order -2.

By construction, we have

$$(\mathscr{L}_2^{\mathfrak{c}})^{-1} = \mathscr{O}_0 \, \mathscr{V} \, (\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r - \mathscr{R})^{-1} \, \mathscr{V}^* \, \mathscr{O}_0^{-1} \,, \qquad \mathscr{R} \in OP\mathcal{H}^{s-1} S^{-4} \,.$$

On the other hand, for large frequencies, we can write

$$\begin{aligned}(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r - \mathscr{R})^{-1} &= (\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{-1} + \sum_{k=1}^{+\infty} \big((\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{-1} \, \mathscr{R}\big)^k \, (\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{-1} \\ &= (\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{-1} + \mathrm{Op}(-4) \,,\end{aligned}$$

and consequently

$$(\mathscr{L}_2^{\mathfrak{c}})^{-1} = \mathscr{O}_0 \, \mathscr{V} \, \big(\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r\big)^{-1} \, \mathscr{V}^* \, \mathscr{O}_0^{-1} + \mathrm{Op}(-4) \,.$$

But on the other hand $\mathscr{V} = \mathrm{Id} + \mathrm{Op}(-3)$. Thus, the non diagonal terms induced by the actions of $\mathscr{V}$ and $\mathscr{V}^*$ can be incorporated in a remainder. More precisely

$$(\mathscr{L}_2^{\mathfrak{c}})^{-1} = \mathscr{O}_0 \begin{pmatrix} \big(\rho(x) + \mathscr{E}_{11}\big)^{-1} & 0 \\ 0 & (\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)_{22}^{-1} \end{pmatrix} \mathscr{O}_0^{-1} + \mathrm{Op}(-3) \,.$$

Now, let $v \in \mathcal{D}^r$. Thus, we have $\mathscr{O}_0^{-1} \, v = {}^t(0, v_2, v_3)$ with $v_j \in \mathcal{H}^r$, so that

$$(\mathscr{L}_2^{\mathfrak{c}})^{-1} \, v = \mathscr{O}_0 \left(\begin{array}{c} 0 \\ ((\mathscr{D}^{\mathfrak{c}} + \mathscr{E}_r)^{Bot}_{2\times 2})^{-1} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} \end{array} \right) + \mathrm{Op}(-3) \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} = \mathrm{Op}(-2) \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} ,$$

which leads to the expected conclusion.

3.2. **Transformation of the compressible equations.** We start by recalling what Lemma 3 becomes in the compressible case.

Lemma 15. *[A decreasing enegy] Let $U(\rho)$ be the internal energy function of the system. It must satisfy $U'(\rho) = \rho^{-2} \, p(\rho) \geq 0$, and it can be adjusted such that $U(0) = 0$ so that $U(\rho) \geq 0$. In particular, for a polytropic equation of state, we find $U(\rho) = c\rho^{\gamma-1}/(\gamma-1)$ where c is a positive constant and $\gamma > 1$ is the heat capacity ratio. Retain that*

$$\mathcal{E}^{\mathfrak{c}}(t) := \frac{1}{2} \int_{\mathbb{R}^3} \Big(\rho \, |v|^2 + 2 \; \rho \, U(\rho) + B \cdot B^* \Big) \, (t, \cdot) \; dx \leq \mathcal{E}^{\mathfrak{c}}(0) \,, \tag{3.12}$$

where

$$\int_{\mathbb{R}^3} B \cdot B^*(t, \cdot) \, dx = \int_{\mathbb{R}^3} \left(|B|^2 + \frac{|\nabla \times B|^2}{\rho} \right) (t, \cdot) \; dx \,.$$

Proof. It is well known, see especially [27] but also [4, 31], that XMHD (with $\nu = 0$) has a Hamiltonian structure conserving the energy $\mathcal{E}^{\mathfrak{c}}$. The inequality inside (3.12) comes from the dissipative effects which are induced by the (fluid) bulk viscosity. Note that there are also (when $\nu = 0$) three independent Casimirs, see (52), (53) and (54) in [1]. ☐

Since $p' > 0$, instead of working with ρ, we can alternatively deal with

$$q := g(\rho) := \int_{\bar\rho}^{\rho} \frac{\sqrt{p'(s)}}{s}\, ds\,, \qquad g'(\rho) = \frac{\sqrt{p'(\rho)}}{\rho} > 0\,, \qquad a(q) := g^{-1}(q)\; g' \circ g^{-1}(q)\,.$$

Expressed in terms of ${}^t(q, v, B^*)$, the system (1.8) becomes symmetric with respect to the two first lines below, that is especially with respect to (q, v). We consider

$$(3.13)\qquad \begin{cases} \partial_t q + (v\cdot\nabla)q + a(q)\ \nabla\cdot v = 0\,, \\ \partial_t v + (v\cdot\nabla)v + a(q)\ \nabla q + B^*\times \mathscr{K}^{\mathfrak{c}}_{-1}\, B^* + \dfrac{1}{2}\ \nabla|\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*|^2 = \nu\ \nabla(\nabla\cdot v)\,, \\ \partial_t B^* + \nabla\times\big(B^*\times(v - d\ \mathscr{K}^{\mathfrak{c}}_{-1}\, B^*)\big) + \nabla\times\big((\nabla\times v)\times\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*\big) = 0\,. \end{cases}$$

In (3.13), in comparison with (1.8), care has been taken to replace everywhere $\nabla\times B$ as prescribed by (3.9). In (3.13), the expression $\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*$ undergoes no more than one derivative. In view of Proposition 14, this means that the corresponding contributions can be seen as acting on B^* like zero order operators (and even of order -1). The equation on B^* contains an inertial contribution at the end of the last line of (3.13), which is of order 2 with respect to v. The idea of Section 2 is to reduce this order to 1 by introducing certain derivatives of v, namely those contained in the vorticity w. Remarkably, the extra derivatives of B^* thus generated (when looking at the equation on w) are exactly balanced inside symmetric structures. In the compressible case, derivatives of q (or ρ) appear during this procedure. Moreover, all derivatives of v are required, including the divergence part $\nabla\cdot v$. Accordingly, we have to introduce the new unknown $U^{\mathfrak{c}}_{\mathfrak{v}} := (q, \nabla q, \nabla\cdot v, w, B^*)$ where $w := \nabla\times v$. From (3.13), we can deduce that

$$(3.14)\qquad \begin{cases} \partial_t q + v\cdot\nabla q + a(q)\ \nabla\cdot v = 0\,, \\ \partial_t(\nabla q) + (v\cdot\nabla)\nabla q + a(q)\ \nabla(\nabla\cdot v) = \mathscr{S}^{\mathfrak{c}\dot q}_{\mathfrak{v}0}\, U^{\mathfrak{c}}_{\mathfrak{v}}\,, \\ \partial_t(\nabla\cdot v) + (v\cdot\nabla)(\nabla\cdot v) + a(q)\ \Delta q - \nu\ \Delta(\nabla\cdot v) = \mathscr{S}^{\mathfrak{c}\dot v}_{\mathfrak{v}1}\, U^{\mathfrak{c}}_{\mathfrak{v}}\,, \\ \partial_t w + v\cdot\nabla w + (\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*\cdot\nabla)B^* = \mathscr{S}^{\mathfrak{c}w}_{\mathfrak{v}0}\, U^{\mathfrak{c}}_{\mathfrak{v}}\,, \\ \partial_t B^* + \big((v - d\,\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*)\cdot\nabla\big)B^* + (\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*\cdot\nabla)w = \mathscr{S}^{\mathfrak{c}B^*}_{\mathfrak{v}0}\, U^{\mathfrak{c}}_{\mathfrak{v}}\,. \end{cases}$$

In (3.14), the velocity v must be deduced from $(w, \nabla\cdot v)$ through the div-curl system (6.4), see Subsection 6.2. On the other hand, the operator $\mathscr{S}^{\mathfrak{c}}_{\mathfrak{v}} = {}^t(0, \mathscr{S}^{\mathfrak{c}\dot q}_{\mathfrak{v}0}, \mathscr{S}^{\mathfrak{c}\dot v}_{\mathfrak{v}0}, \mathscr{S}^{\mathfrak{c}w}_{\mathfrak{v}0}, \mathscr{S}^{\mathfrak{c}B^*}_{\mathfrak{v}0})$ put in source term is outlined below

$$(3.15)\qquad \begin{aligned} \mathscr{S}^{\mathfrak{c}\dot q}_{\mathfrak{v}0}\, U^{\mathfrak{c}}_{\mathfrak{v}} &:= -\nabla q\ Dv - a'(q)\ (\nabla\cdot v)\ \nabla q\,, \\ \mathscr{S}^{\mathfrak{c}\dot v}_{\mathfrak{v}1}\, U^{\mathfrak{c}}_{\mathfrak{v}} &:= -\sum_{i=1}^{3}(\partial_i v\cdot\nabla)v_i - a'(q)\ |\nabla q|^2 - \nabla\cdot(\mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0}\, U^{\mathfrak{c}}_{\mathfrak{v}})\,, \\ \mathscr{S}^{\mathfrak{c}w}_{\mathfrak{v}0}\, U^{\mathfrak{c}}_{\mathfrak{v}} &:= (B^*\cdot\nabla)(\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*) + (\nabla\cdot B^*_0)\ \mathscr{K}^{\mathfrak{c}}_{-1}\, B^* \\ &\quad\ - \nabla\cdot(\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*)\ B^* + (w\cdot\nabla)v - w\ (\nabla\cdot v)\,, \\ \mathscr{S}^{\mathfrak{c}B^*}_{\mathfrak{v}0}\, U^{\mathfrak{c}}_{\mathfrak{v}} &:= -\big(\nabla\cdot v - d\ (\nabla\cdot\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*)\big)\ B^* + (\nabla\cdot B^*_0)\ (v - d\ \mathscr{K}^{\mathfrak{c}}_{-1}\, B^*) \\ &\quad\ + (B^*\cdot\nabla)(v - d\ \mathscr{K}^{\mathfrak{c}}_{-1}\, B^*) - \nabla\cdot(\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*)\ w + (w\cdot\nabla)(\mathscr{K}^{\mathfrak{c}}_{-1}\, B^*)\,, \end{aligned}$$

where $\mathscr{S}^{\mathfrak{c}\dot{v}}_{\mathfrak{v}1}$ is indeed of order 1 since $\mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0}\,U^{\mathfrak{c}}_{\mathfrak{v}} := B^*\times\mathscr{K}^{\mathfrak{c}}_{-1}\,B^*+\frac{1}{2}\,\nabla|\mathscr{K}^{\mathfrak{c}}_{-1}\,B^*|^2$. In comparison with $\mathscr{S}^{\mathfrak{i}}_{\mathfrak{v}0}\,U^{\mathfrak{i}}_{\mathfrak{v}}$, we have many new contributions. Some of them come from the fact that $\mathscr{K}^{\mathfrak{c}}_{-1}\,B^*$ and B^* are no more solenoidal vector fields. Observe that we have exploited Lemma 3.1 to replace $\nabla\cdot B^*$ everywhere by $\nabla\cdot B^*_0$ (this is essential to avoid artificial losses of derivatives coming from $\nabla\cdot B^*$). We have also incorporated the vortex stretching induced by $\nabla\cdot v$ (which is no more zero) as well as other contributions related to $\nabla\cdot v$.

3.3. **Proof of Theorem 2.** We start by showing Theorem 2 under the more restrictive regularity assumption $s>7/2$. We refer to Section 4 for the optimal result. From (1.20), we know that $\rho_0\in\mathcal{H}^s$, and by construction we have $q_0=g(\bar\rho+\rho_0)$ with $g(\bar\rho)=0$. Then, by the rule of composition in $\mathcal{H}^s$, we recover that $q_0\in\mathcal{H}^s$. In particular, at time $t=0$, with $\tilde s:=s-1>5/2$, we can assert that

$$U^{\mathfrak{c}}_{\mathfrak{v}}(0,\cdot)=U^{\mathfrak{c}}_{\mathfrak{v}0}=(q_0,\nabla q_0,\nabla\cdot v_0,w_0,B^*_0)\in\mathcal{H}^{\tilde s}\,,\qquad w_0:=\nabla\times v_0\,. \tag{3.16}$$

Any smooth solution to (1.8)-(1.9)-(1.10) leads to a solution to (3.14)-(3.16), and conversely. We study below the time evolution of the L^2-norm of $U^{\mathfrak{c}}_{\mathfrak{v}}$ (assuming for the moment that $U^{\mathfrak{c}}_{\mathfrak{v}}$ is bounded in the large $\mathcal{H}^{\tilde s}$-norm).

Lemma 16. *[L^2-energy estimate for the vorticity formulation in the compressible case] Let $T>0$. Assume that $U^{\mathfrak{c}}_{\mathfrak{v}}\in C([0,T];\mathcal{H}^{\tilde s})$ is a solution to (3.14) with initial data (3.16). Then, we can find a constant C depending only on the $C([0,T];\mathcal{H}^{\tilde s})$-norm of $U^{\mathfrak{c}}_{\mathfrak{v}}$ such that*

$$\parallel U^{\mathfrak{c}}_{\mathfrak{v}}(t,\cdot)\parallel_{L^2}\le\parallel U^{\mathfrak{c}}_{\mathfrak{v}0}\parallel_{L^2}\; e^{C\,t+C\,t/\nu}\,,\quad \forall\,t\in[0,T]\,. \tag{3.17}$$

Proof. Multiply (3.14) by ${}^tU^{\mathfrak{c}}_{\mathfrak{v}}$, and integrate with respect to x. After integrations by parts

$$\begin{aligned}&\frac12\,\frac{d}{dt}\Big(\int_{\mathbb{R}^3}|U^{\mathfrak{c}}_{\mathfrak{v}}(t,\cdot)|^2\,dx\Big)+\nu\int_{\mathbb{R}^3}|\nabla(\nabla\cdot v)|^2\,dx\\ &\quad=\int_{\mathbb{R}^3}(\nabla q\cdot\mathscr{S}^{\mathfrak{c}\dot q}_{\mathfrak{v}0}\,U^{\mathfrak{c}}_{\mathfrak{v}}+(\nabla\cdot v)\;\mathscr{S}^{\mathfrak{c}\dot v}_{\mathfrak{v}1}\,U^{\mathfrak{c}}_{\mathfrak{v}}+w\cdot\mathscr{S}^{\mathfrak{c}w}_{\mathfrak{v}0}\,U^{\mathfrak{c}}_{\mathfrak{v}}+B^*\cdot\mathscr{S}^{\mathfrak{c}B^*}_{\mathfrak{v}0}\,U^{\mathfrak{c}}_{\mathfrak{v}})\;dx\\ &\quad\quad+\int_{\mathbb{R}^3}(h_0+h_1+h_2+h_3+h_4)\;dx\,,\end{aligned} \tag{3.18}$$

where the $\mathscr{S}^{\mathfrak{c}\star}_{\mathfrak{v}\diamond}$ are the source terms of (3.15), whereas the $h_\star$ come from the quasilinear parts. We find that

$$\begin{aligned}&\int_{\mathbb{R}^3}h_0\;dx:=-\int q\;a(q)\;(\nabla\cdot v)\;dx\,,\\ &\int_{\mathbb{R}^3}h_1\;dx:=-\int U^{\mathfrak{c}}_{\mathfrak{v}}\cdot((v\cdot\nabla)U^{\mathfrak{c}}_{\mathfrak{v}})\;dx=\frac12\int(\nabla\cdot v)\;|U^{\mathfrak{c}}_{\mathfrak{v}}|^2\;dx\,,\\ &\int_{\mathbb{R}^3}h_2\;dx:=-\int_{\mathbb{R}^3}a(q)\;(\nabla q\cdot\nabla(\nabla\cdot v)+(\nabla\cdot v)\;\Delta q)\;dx=\int_{\mathbb{R}^3}a'(q)\;(\nabla\cdot v)\;|\nabla q|^2\;dx\,,\\ &\int_{\mathbb{R}^3}h_3\;dx:=\int_{\mathbb{R}^3}(\nabla\cdot(\mathscr{K}^{\mathfrak{c}}_{-1}\,B^*))\;(w\cdot B^*)\;dx\,,\\ &\int_{\mathbb{R}^3}h_4\;dx:=-\frac{d}{2}\int_{\mathbb{R}^3}(\nabla\cdot(\mathscr{K}^{\mathfrak{c}}_{-1}\,B^*))\;|B^*|^2\;dx\,.\end{aligned}$$

We consider each term separately. Knowing that $\tilde{s} > 5/2$, we use repeatedly the Sobolev embedding theorem $\mathcal{H}^{\tilde{s}} \hookrightarrow L^\infty$. We also exploit Proposition 14 with $r = s-2$ to deduce from $\nabla \times B^* \in \mathcal{H}^{\tilde{s}-1} \equiv \mathcal{H}^{s-2}$ that $\mathscr{K}^{\mathfrak{c}}_{-1} B^* = (\mathscr{L}^{\mathfrak{c}}_2)^{-1} \nabla \times B^* \in \mathcal{H}^s$. In other words

$$\| \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{\mathcal{H}^{\tilde{s}}} \lesssim \| B^* \|_{\mathcal{H}^{\tilde{s}-1}} , \qquad \| \nabla \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{\mathcal{H}^{\tilde{s}}} \lesssim \| B^* \|_{\mathcal{H}^{\tilde{s}}} . \tag{3.19}$$

On the other hand, by assumption, we know that

$$\| U^{\mathfrak{c}}_{\mathfrak{v}} \|_{L^\infty([0,T]\times\mathbb{R}^3)} + \| \nabla U^{\mathfrak{c}}_{\mathfrak{v}} \|_{L^\infty([0,T]\times\mathbb{R}^3)} \lesssim M := \| U^{\mathfrak{c}}_{\mathfrak{v}} \|_{C([0,T];\mathcal{H}^{\tilde{s}})} < +\infty .$$

First, from Lemma 24, we have

$$\left| \int_{\mathbb{R}^3} \nabla q \cdot \mathscr{S}^{\mathfrak{c}\dot{q}}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}} \, dx \right| \leq \big(\| \nabla q \|_{L^\infty} + \| \nabla a(q) \|_{L^\infty} \big) \ \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} \lesssim \ M \ \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} .$$

We now turn to the equation on $\nabla \cdot v$, where the source term $\mathscr{S}^{\mathfrak{c}\dot{v}}_{\mathfrak{v}1} U^{\mathfrak{c}}_{\mathfrak{v}}$ does include a loss of one derivative. The idea is to compensate this after integration by parts by the bulk (fluid) viscosity. With the help of Lemma 24, this gives rise to

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\nabla \cdot v) \ \mathscr{S}^{\mathfrak{c}\dot{v}}_{\mathfrak{v}1} U^{\mathfrak{c}}_{\mathfrak{v}} \, dx \right| \leq & \sum_{i=1}^3 \int_{\mathbb{R}^3} |\nabla \cdot v| \ |\partial_i v| \ |\nabla v_i| \ dx + \int_{\mathbb{R}^3} |\nabla \cdot v| \ |a'(q)| \ |\nabla q|^2 \ dx \\ & + \int_{\mathbb{R}^3} |\nabla(\nabla \cdot v)| \ |\mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}| \ dx \\ \leq & \ C \ (\| U^{\mathfrak{c}}_{\mathfrak{v}} \|_{L^\infty}) \ \ \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} \\ & + c \ \nu \int_{\mathbb{R}^3} |\nabla(\nabla \cdot v)|^2 \ dx + \frac{C}{\nu} \int_{\mathbb{R}^3} |\mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}|^2 \ dx , \end{aligned}$$

where the constant $c \in \mathbb{R}^*_+$ can be chosen as small as wished. Observe that

$$\int_{\mathbb{R}^3} |\mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}|^2 \ dx \leq \| \mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}} \|_{L^\infty} \int_{\mathbb{R}^3} |\mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}| \ dx ,$$

From (3.19), we have

$$\begin{aligned} \| \mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}} \|_{L^\infty} & \lesssim \| \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{L^\infty} \ \big(\| B^* \|_{L^\infty} + \| \nabla \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{L^\infty} \big) \\ & \lesssim \| \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{\mathcal{H}^{\tilde{s}}} \ \big(\| B^* \|_{\mathcal{H}^{\tilde{s}}} + \| \nabla \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{\mathcal{H}^{\tilde{s}}} \big) \lesssim \| B^* \|^2_{\mathcal{H}^{\tilde{s}}} \lesssim M^2 . \end{aligned}$$

On the other hand

$$\int_{\mathbb{R}^3} |\mathscr{S}^{\mathfrak{c}v}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}| \ dx \leq \| \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{L^2} \ \big(\| B^* \|_{L^2} + \| \nabla \mathscr{K}^{\mathfrak{c}}_{-1} B^* \|_{L^2} \big) \lesssim \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} .$$

Thus, we can retain that

$$\left| \int_{\mathbb{R}^3} (\nabla \cdot v) \ \mathscr{S}^{\mathfrak{c}\dot{v}}_{\mathfrak{v}1} U^{\mathfrak{c}}_{\mathfrak{v}} \, dx \right| \leq C \ M \ \Big(1 + \frac{M}{\nu} \Big) \ \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} + c \ \nu \int_{\mathbb{R}^3} |\nabla(\nabla \cdot v)|^2 \ dx .$$

Thanks to Lemma 11 and (3.19), the two contributions $\mathscr{S}^{\mathfrak{c}w}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}$ and $\mathscr{S}^{\mathfrak{c}B^*}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}$ are both of order 0 in terms of $U^{\mathfrak{c}}_{\mathfrak{v}}$. Be careful, in the two definitions of $\mathscr{S}^{\mathfrak{c}w}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}$ and $\mathscr{S}^{\mathfrak{c}B^*}_{\mathfrak{v}0} U^{\mathfrak{c}}_{\mathfrak{v}}$ some derivatives act on $\mathscr{K}^{\mathfrak{c}}_{-1}$ and therefore on ρ (since the operator $\mathscr{K}^{\mathfrak{c}}_{-1}$ involves coefficients depending on ρ). But these one order derivatives of ρ are included in $U^{\mathfrak{c}}_{\mathfrak{v}}$ (through the derivatives of q), and therefore this corresponds indeed to zero order contributions.

As described above, we can obtain

$$\Big| \int_{\mathbb{R}^3} w \cdot \mathscr{S}^{\mathfrak{c} w}_{\mathfrak{v}0} \, U^{\mathfrak{c}}_{\mathfrak{v}} \, dx \Big| \lesssim M \ \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} \, , \qquad \Big| \int_{\mathbb{R}^3} B^* \cdot \mathscr{S}^{\mathfrak{c} B^*}_{\mathfrak{v}0} \, U^{\mathfrak{c}}_{\mathfrak{v}} \, dx \Big| \lesssim M \ \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} \, .$$

The same sort of arguments applies to handle the $h_\star$. We find that

$$\Big| \int_{\mathbb{R}^3} h_j \, dx \Big| \lesssim M \ \| U^{\mathfrak{c}}_{\mathfrak{v}} \|^2_{L^2} \, , \qquad \forall \, j \in \{0, \cdots, 4\} \, .$$

By selecting c small enough, we can absorb the term implying the L^2-norm of $\nabla(\nabla \cdot v)$. Note that the presence of a bulk (fluid) viscosity is crucial here to compensate for losses related to $\nabla(\nabla \cdot v)$. At the end, there remains

$$\frac{d}{dt} \Big(\int_{\mathbb{R}^3} |U^{\mathfrak{c}}_{\mathfrak{v}}(t,\cdot)|^2 \, dx \Big) \leq C \ M \ \Big(1 + \frac{M}{\nu} \Big) \int_{\mathbb{R}^3} |U^{\mathfrak{c}}_{\mathfrak{v}}(t,\cdot)|^2 \, dx \, .$$

It suffices to implement Grönwall's inequality to recover the inequality (3.17) for some convenient constant C. □

Starting from there, the construction of $\mathcal{H}^{\tilde{s}}$-solutions to (3.14) can be achieved through the general strategy [6, 33] already explained at the end of Section 2. Note that the lifespan T does depend on the bulk viscosity ν. It shrinks to 0 when ν goes to $0+$. Then, from the $\mathcal{H}^{\tilde{s}}$-solution to (3.14), we can recover solutions to (1.8) which are such that $(q, v, B^*) \in \mathcal{H}^s \times \mathcal{H}^s \times \mathcal{H}^{s-1}$ with $s > 7/2$. Starting from there and from (3.9), we get that $\nabla \times B \in \mathcal{H}^s$, while $\nabla \cdot B = \nabla \cdot B^*_0 \in \mathcal{H}^s$. It follows that $B \in \mathcal{H}^{s+1}$ as indicated. This concludes the proof of Theorem 2 at least when $s > 7/2$. The case $s > 5/2$ is investigated in the next section.

4. The potential formulations

The aim of this section is to develop an alternative to the vorticity formulations. The basic idea is to integrate B^* instead of looking at derivatives of (q, v). From $\mathscr{P} B^*$, or just from B^* when $\nabla \cdot B^* = 0$, we can extract a *magnetic potential* A^* which is defined by

$$\nabla \times A^* = \mathscr{P} B^* \, , \qquad \nabla \cdot A^* = 0 \, . \tag{4.1}$$

From now on, we imply ρ, v and A^* as new unknowns, with the following motivations:

- *A simplified presentation.* Like in (1.8), the (compressible) potential formulation involves 7 unknowns, namely the components of ρ, v and A^*. Contrary to (3.14), there is no need of introducing four supplementary unknowns.
- *A better regularity result.* Up to now, we have worked with $s > 7/2$. As stated in Theorems 1 and 2, we would like to improve the threshold up to $s > 5/2$. To this end, we will avoid the use of Proposition 14 which is costly in terms of the regularity of ρ.
- *Additional insights.* This change of point of view offers complementary perspectives. For instance, in the compressible case, it allows to better understand why a bulk (fluid) viscosity is required for stability.

The potential formulations are therefore more simple in appearance and more efficient in some respects. However, there are important subtleties when performing L^2-energy estimates, explaining why this approach has been postponed until now. On the other hand, the vorticity formulations are necessary and instructive to understand how higher order energy estimates can be performed at the level of the potential formulations.

From (3.2) and (4.1), we deduce that $\partial_t B^* = \nabla \times \partial_t A^*$. It follows that the curl operator can be put in factor of the last equation of (1.8), where it can be cancelled (modulo a gradient). This idea applies quite directly in the incompressible context of Subsection 4.1. It must be carefully implemented in the compressible framework of Subsection 4.2.

4.1. **The incompressible situation.** The condition $\nabla \cdot B^* = 0$ and the constant density make things easier. In Paragraph 4.1.1, we derive the incompressible potential equations. In Paragraph 4.1.2, we perform L^2-energy estimates on linearized equations. In Paragraph 4.1.3, we conclude the proof of Theorem 1.

4.1.1. *The incompressible potential equations.* The unknown is $U^{\mathfrak{i}}_{\mathfrak{p}} := (v, A^*)$. We consider the system

$$\begin{cases} \partial_t v + (v\cdot\nabla)v - (A^* - A)\times(\nabla\times A^*) + \nabla p = 0\,, \\ \partial_t A^* - (v - d\,(A^* - A))\times(\nabla\times A^*) - (A^* - A)\times(\nabla\times v) + \nabla e = 0\,, \end{cases} \tag{4.2}$$

where both v and A^* are solenoidal vector fields

$$\nabla\cdot v = 0\,, \qquad \nabla\cdot A^* = 0\,. \tag{4.3}$$

while A can be obtained from A^* through

$$A = (\mathrm{Id} - \Delta)^{-1} A^*\,, \qquad \nabla\cdot A = 0\,, \tag{4.4}$$

The introduction of the Lagrange multipliers (scalar functions) p and e is needed above to ensure the propagation of the constraints $\nabla\cdot v = 0$ and $\nabla\cdot A^* = 0$.

Lemma 17. *[Link between the incompressible potential and vorticity formulations] Let $U^{\mathfrak{i}}_{\mathfrak{p}} = (v, A^*)$ be some $\mathcal{H}^s$-solution on $[0,T]$ to (4.2)-(4.4)-(4.3) with $s > 5/2$. Define*

$$B^* := \nabla\times A^*\,, \qquad B := \nabla\times A\,. \tag{4.5}$$

Then, (v, B^*, B) is a solution on $[0,T]$ to (1.15)-(1.16)-(1.17), which is as in (1.19).

Proof. By construction, we have (1.15). On the other hand, by applying the curl operator to (4.4), we get (1.17). Since $\nabla\cdot A = 0$, the relation (4.4) is the same as

$$A^* - A = -\Delta A = \nabla\times(\nabla\times A) = \nabla\times B\,.$$

It follows that

$$-(A^* - A)\times(\nabla\times A^*) = B^*\times(\nabla\times B)\,.$$

Exploiting the two above relations and applying the curl operator $\nabla\times$ to the second line of (4.2), the term ∇e disappears and we have directly access to (1.16). □

At the initial time $t = 0$, we impose

$$U^{\mathfrak{i}}_{\mathfrak{p}}(0,\cdot) = U^{\mathfrak{i}}_{\mathfrak{p}0} = (v_0, A^*_0) \in \mathcal{D}^s(\mathbb{R}^3;\mathbb{R}^3)^2\,, \qquad s > 5/2\,. \tag{4.6}$$

4.1.2. *L^2-energy estimates.* The conserved quantity $\mathcal{E}^{\mathfrak{i}}$ of Lemma 3, see the definition (2.1), can be reformulated according to

$$\mathcal{E}^{\mathfrak{i}} := \frac{1}{2}\int_{\mathbb{R}^3} \big(|v|^2 + |\nabla \times (\mathrm{Id}-\Delta)^{-1}A^*|^2 + |\Delta\,(\mathrm{Id}-\Delta)^{-1}A^*|^2\big)\, dx < +\infty\,. \tag{4.7}$$

This already furnishes some (high frequency) L^2-bound concerning $U^{\mathfrak{i}}_{\mathfrak{p}}$. But this is not enough. To construct solutions by a fixed point argument, we also need to consider the stability issue. To this end, we have to look at the linearized equations coming from (4.2), dealing with $\dot U^{\mathfrak{i}}_{\mathfrak{p}} = (\dot v, \dot A^*)$. When doing this, the term which for instance is at top right of (4.2) leads to $-(A^*-A)\times(\nabla\times\dot A^*) + (\nabla\times A^*)\times\big(\dot A^* - (\mathrm{Id}-\Delta)^{-1}\dot A^*\big)$. Given a Lipschitz field A^*, the right hand side is such that

$$\parallel (\nabla\times A^*)\times\big(\dot A^* - (\mathrm{Id}-\Delta)^{-1}\dot A^*\big) \parallel_{L^2} \lesssim \parallel \dot U^{\mathfrak{i}}_{\mathfrak{p}} \parallel_{L^2}\ .$$

Such contributions clearly cannot undermine the local L^2-stability. Thus, to simplify the presentation, they can be ignored. We can focus on

$$\left\{\begin{array}{l} \partial_t \dot v + (v\cdot\nabla)\dot v + \nabla\dot p - (A^*-A)\times(\nabla\times\dot A^*) = 0\,, \\ \partial_t \dot A^* - \big(v - d\,(A^*-A)\big)\times(\nabla\times\dot A^*) - (A^*-A)\times(\nabla\times\dot v) + \nabla\dot e = 0\,, \end{array}\right. \tag{4.8}$$

together with

$$\nabla\cdot\dot v = 0\,, \qquad \nabla\cdot\dot A^* = 0\,. \tag{4.9}$$

At the initial time $t=0$, we impose

$$\dot U^{\mathfrak{i}}_{\mathfrak{p}}(0,\cdot) = \dot U^{\mathfrak{i}}_{\mathfrak{p}0} = (\dot v_0, \dot A^*_0) \in \mathcal{D}^0(\mathbb{R}^3;\mathbb{R}^3)^2\,. \tag{4.10}$$

Lemma 18. *[L^2-energy estimates for the linearized incompressible potential equations] Let $T>0$. Assume that $U^{\mathfrak{i}}_{\mathfrak{p}} = (v, A^*)$ is such that $U^{\mathfrak{i}}_{\mathfrak{p}} \in C([0,T];\mathcal{D}^s)$ for some $s>5/2$. Then, the Cauchy problem (4.8)-(4.9) with initial data (4.10) has a solution on $[0,T]$. Moreover, we can find a constant C depending only on the $C([0,T];\mathcal{H}^s)$-norm of $U^{\mathfrak{i}}_{\mathfrak{p}}$ such that*

$$\parallel \dot U^{\mathfrak{i}}_{\mathfrak{p}}(t,\cdot) \parallel_{L^2} \leq \parallel \dot U^{\mathfrak{i}}_{\mathfrak{p}0} \parallel_{L^2}\ e^{C\,t}\,, \qquad \forall\, t\in[0,T]\,. \tag{4.11}$$

Any $\mathcal{H}^s$-solution to the initial value problem (4.2)-(4.4)-(4.3)-(4.6) leads to a solution to (4.8)-(4.9) with initial data $\dot U^{\mathfrak{i}}_{\mathfrak{p}0} = U^{\mathfrak{i}}_{\mathfrak{p}0}$. As a consequence, the proof of Lemma 18 gives another access to some L^2-bound, namely

$$\parallel U^{\mathfrak{i}}_{\mathfrak{p}}(t,\cdot) \parallel_{L^2} \leq \parallel U^{\mathfrak{i}}_{\mathfrak{p}0} \parallel_{L^2}\ e^{C\,t}\,, \qquad \forall\, t\in[0,T]\,.$$

Proof. To gain a better grasp of the arguments, we have made a clear distinction between two kinds of quantities:

- On the one hand, there are those which play the role of coefficients and which are managed through the assumption

$$\parallel U^{\mathfrak{i}}_{\mathfrak{p}} \parallel_{L^\infty([0,T]\times\mathbb{R}^3)} + \parallel \nabla U^{\mathfrak{i}}_{\mathfrak{p}} \parallel_{L^\infty([0,T]\times\mathbb{R}^3)} \lesssim M := \parallel U^{\mathfrak{i}}_{\mathfrak{p}} \parallel_{C([0,T];\mathcal{H}^s)} < +\infty\,. \tag{4.12}$$

- On the other hand, there are those which are handled as unknowns and which are marked by a dot, like $\dot U^{\mathfrak{i}}_{\mathfrak{p}} = (\dot v, \dot A^*)$.

We now exploit the formalism of Remark 12. We denote by $\mathscr{T}_C$ with $C = A^* - A$ or $C = v$ the operator defined at the level of (3.3). We find that $\mathscr{T}_C^* \dot A^* = -C \times (\nabla \times \dot A^*)$. Thus, the system (4.8) can be rewritten according to

$$\left\{ \begin{array}{l} \partial_t \dot v + (v\cdot\nabla)\dot v + \nabla \dot p + \mathscr{T}^*_{A^*-A} \dot A^* = 0\,, \\ \partial_t \dot A^* + \mathscr{T}^*_v \dot A^* - d\ \mathscr{T}^*_{A^*-A}\dot A^* + \mathscr{T}^*_{A^*-A}\dot v + \nabla \dot e = 0\,. \end{array}\right. \tag{4.13}$$

As already noted, the operator $\mathscr{T}_C$ is not skew-adjoint and, of course, neither is $\mathscr{T}_C^*$. But (Remark 12), knowing that the contribution $\nabla\cdot B^*$ is given (or can be forgotten), the action of $\mathscr{T}_C$ (viewed as $\tilde{\mathscr{T}}_C$) on B^* becomes skew-adjoint. This argument was crucial in Sections 2 and 3. None of that applies to the action of $\mathscr{T}_C^*$ on $\dot A^*$ (because the analogue of $\nabla\cdot B^* = 0$ in the context of $\mathscr{T}_C^* \dot A^*$ is not $\nabla\cdot \dot A^* = 0$). In other words, (4.13) is not well-posed, while its dual version is, in the sense that it becomes symmetric under the condition (4.9). To put this principle into practice, we multiply the first and second equation of (4.13) respectively by $\dot v$ and $\dot A^*$; we integrate with respect to the variable x; and then we force the emergence of the operator $\mathscr{T}_C$ (instead of $\mathscr{T}_C^*$) by passing to the adjoint. Since $v\in\mathcal{D}^s$, the contribution related to $v\cdot\nabla$ disappears. Since $\nabla\cdot\dot A^* = 0$, the term involving $\nabla\dot e$ is eliminated. Denoting by $\langle\cdot,\cdot\rangle$ the scalar product in L^2, this furnishes

$$\frac12\ \frac{d}{dt}\Big(\int_{\mathbb{R}^3} |\dot U^{\mathfrak{i}}_{\mathfrak{p}}(t,\cdot)|^2\ dx\Big) = -\langle \mathscr{T}_{A^*-A}\dot v, \dot A^*\rangle - \langle \mathscr{T}_v \dot A^*, \dot A^*\rangle + \langle \mathscr{T}_{A^*-A}\dot A^*, d\,\dot A^* - \dot v\rangle\,.$$

Observe the changeover from $\mathscr{T}_C^*$ to $\mathscr{T}_C$. From there, the decomposition (3.5) becomes pertinent. In the actual incompressible situation, using (6.2), this yields

$$\begin{aligned} \frac12\ \frac{d}{dt}\Big(\int_{\mathbb{R}^3} |\dot U^{\mathfrak{i}}_{\mathfrak{p}}(t,\cdot)|^2\ dx\Big) = {} & \frac12 \int_{\mathbb{R}^3} \Big(\big((A^*-A)\cdot\nabla\big)\big(\dot A^*\cdot(2\,\dot v - d\,\dot A^*)\big) + (v\cdot\nabla)|\dot A^*|^2\Big)\ dx \\ & - \langle (\dot v\cdot\nabla)(A^*-A), \dot A^*\rangle - \langle (\dot A^*\cdot\nabla)v, \dot A^*\rangle \\ & + \langle (\dot A^*\cdot\nabla)(A^*-A), d\,\dot A^* - \dot v\rangle\,. \end{aligned}$$

In the right hand side, since A^*-A and v are solenoidal vector fields, after integration by parts, the first line disappears. Exploiting (4.12) to control the two last lines, we find that

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^3} |\dot U^{\mathfrak{i}}_{\mathfrak{p}}(t,\cdot)|^2\ dx\Big) \lesssim \int_{\mathbb{R}^3} |\dot U^{\mathfrak{i}}_{\mathfrak{p}}(t,\cdot)|^2\ dx\,.$$

By Grönwall's inequality, we recover (4.11). □

4.1.3. *End of the proof of Theorem 1.* The construction of $\mathcal{H}^s$-solutions to (4.2) follows the lines mentioned before (it is not detailed). But, we add a few words about how higher order estimates can be obtained. In the context of (4.2), there are two ways to proceed:

- The incompressible vorticity formulation is in fact similar to a derived version of the incompressible potential formulation. It follows that the preceding L^2-estimates for (2.11)-(2.12) correspond to one order estimates for (4.2)-(4.4). In other words, the work of Subsection 2.4 can be seen as the first stage to check that higher order estimates (namely $\mathcal{H}^1$-estimates) are available for the incompressible potential formulation.

- Looking at the linearized equations (4.13) with adequate source terms amounts to the same thing as studying the equations satisfied by the derivatives of $U^{\mathfrak{i}}_{\mathfrak{p}}$. Thus, the proof of Lemma 9 already provides with convincing points of reference towards $\mathcal{H}^s$-estimates.

4.2. **The compressible framework.** The general lines are as in Subsection 4.1 but the variations of ρ oblige to adapt a number of aspects. The first step (in Paragraph 4.2.1) is to correctly interpret (1.9) in terms of the potentials A^* and A. The second stage (in Paragraph 4.2.2) is to propose potential equations that are compatible with (1.8). Then (in Paragraph 4.2.3), we explain how to obtain L^2-energy estimates on linearized equations.

4.2.1. *The potential constitutive relation.* Keeping (4.1) and assuming that $\mathscr{P}B = \nabla \times A$, the constitutive relation (1.9) is the same as

$$\mathscr{P}B^* - \mathscr{P}B - \nabla \times \Big(\frac{\nabla \times B}{\rho(x)} \Big) = \nabla \times \Big(A^* - A - \frac{\nabla \times B}{\rho(x)} \Big) = 0 \, .$$

This suggests to impose

$$A^* - A - \frac{\nabla \times B}{\rho(x)} = 0 \, . \tag{4.14}$$

Exploiting (3.7) with again (4.5), this is equivalent to

$$A = A^* - (\mathscr{L}^{\mathfrak{c}}_2)^{-1} \, \nabla \times (\nabla \times A^*) = (\mathscr{L}^{\mathfrak{c}}_2)^{-1} (\rho \, A^*) \, . \tag{4.15}$$

By this way, A is deduced from A^* through a pseudo-differential operator which is of order zero (or less). In what follows, we do not need Proposition 14. Instead, we are satisfied with Lemma 13 leading to

$$\parallel A \parallel_{L^2} \lesssim \parallel A^* \parallel_{L^2} \ . \tag{4.16}$$

One of the difficulties is to show that the choice (4.15) is appropriate.

4.2.2. *The compressible potential equations.* Let $p : \mathbb{R} \to \mathbb{R}$ be any smooth function (of ρ). Fix some initial data (ρ_0, v_0, B^*_0) as in (1.20). From B^*_0, extract the vector field A^*_0 which is such that

$$\nabla \times A^*_0 = \mathscr{P} \, B^*_0 \, , \qquad \nabla \cdot A^*_0 = 0 \, , \qquad A^*_0 \in \mathcal{H}^s \, .$$

The unknown is $U^{\mathfrak{c}}_{\mathfrak{p}} := (\rho, v, A^*)$. Consider the system

$$\left\{ \begin{aligned} & \partial_t \rho + (v \cdot \nabla) \rho + \rho \, \nabla \cdot v = 0 \, , \\ & \partial_t v + (v \cdot \nabla) v + \frac{\nabla p}{\rho} - (A^* - A) \times (\nabla \times A^*) \\ & \qquad \qquad + \nabla (|A^* - A|^2/2) = \nu \ \nabla (\nabla \cdot v) + (A^* - A) \times \mathscr{Q} B^*_0 \, , \\ & \partial_t A^* - (v - d \, (A^* - A)) \times (\nabla \times A^*) - (A^* - A) \times (\nabla \times v) + \nabla e \\ & \qquad \qquad \qquad = \big(v - d \, (A^* - A) \big) \times \mathscr{Q} B^*_0 \, , \end{aligned} \right. \tag{4.17}$$

together with

$$\nabla \cdot A^* = 0 \, , \tag{4.18}$$

where A is deduced from A^* through (4.15). At the initial time $t = 0$, we impose

$$U^{\mathfrak{c}}_{\mathfrak{p}}(0,\cdot) = U^{\mathfrak{c}}_{\mathfrak{p}0} = (\rho_0, v_0, A^*_0) \in \mathcal{H}^s(\mathbb{R}^3;\mathbb{R}) \times \mathcal{H}^s(\mathbb{R}^3;\mathbb{R}^3)^2\,, \qquad s > 5/2\,. \tag{4.19}$$

Lemma 19. *[Link between the compressible potential and vorticity formulations] Let $U^{\mathfrak{c}}_{\mathfrak{p}}$ be some $\mathcal{H}^s$-solution on $[0,T]$ to (4.15)-(4.17) with initial data as in (4.19). Define*

$$B^* := \mathscr{Q} B^*_0 + \nabla \times A^*\,, \qquad B := \mathscr{Q} B^*_0 + \nabla \times A\,. \tag{4.20}$$

Then, (ρ, v, B^*, B) is a solution on $[0,T]$ to (1.8)-(1.9), which is associated with the initial data (ρ_0, v_0, B^*_0), and which is as in (1.21).

Note that the solution to (4.17) is no more subjected to $\nabla \cdot v = 0$, but we have still $\nabla \cdot A^* = 0$. The part $\mathscr{Q} A$ is not involved at the level of (4.20), though it is specified when solving (4.15). In view of (4.14), in general, we do not have $\nabla \cdot A = 0$.

Proof. By construction, we have $B^*(0,\cdot) = B^*_0$, and therefore $(\rho, v, B^*)(0,\cdot) = (\rho_0, v_0, B^*_0)$ as required. On the other hand, it is clear that (ρ, v, B^*) is as indicated in (1.21). Let us consider B. Since $\rho\, A^* \in \mathcal{H}^s$, from (4.15), we get that $A \in \mathcal{H}^{s+2}$. Then, from (4.20), we can deduce that $\nabla \times B \in \mathcal{H}^s$, while by assumption $\nabla \cdot B = \nabla \cdot B^*_0 \in \mathcal{H}^s$. Thus, we find that $B \in \mathcal{H}^{s+1}$ as claimed at the level of (1.21). From (4.20), we find that $B^* - B = \nabla \times A^* - \nabla \times A$. Then, by applying the curl operator to (4.14), we obtain (1.9). The equation on ρ is unchanged. In view of (4.14) and (4.20), the equation on v inside (4.17) is just a rephrasing of the equation on v inherited from (1.8). This also applies to the last equation of (4.17) after applying the curl operator to it. □

The equations inside (4.17) bear some similarity to symmetric hyperbolic-parabolic systems which can be put in a normal form in the sense of Kawashima-Shizuta [25]. To see why, we have to check that the conditions enumerated in Section 3 of [25] do apply (at least formally). To match with the notations of [25], define $v_I := {}^t(\rho, \mathscr{P} v, \mathscr{P} A^*)$ and $v_{II} := \mathscr{Q} v$. This repartition gives rise to

$$\left\{ \begin{array}{l} \partial_t v_I + A_I(v_I, v_{II}, D_x) v_I = \bar{g}_I(v_I, v_{II}, D_x v_{II})\,, \\ \partial_t v_{II} = \nu\ \nabla(\nabla \cdot v_{II}) + \bar{g}_{II}(v_I, v_{II}, D_x v_I, D_x v_{II})\,, \end{array} \right. \tag{4.21}$$

where $\bar{g}_I = \bar{g}^1_I(v_I, v_{II}, D_x v_{II}) + \bar{g}^2_I(v_I, v_{II})$ and

$$A_I := \begin{pmatrix} v\cdot\nabla & 0 & 0 \\ 0 & \mathscr{P}\,(v\cdot\nabla)\,\mathscr{P} & \mathscr{P}\,\mathscr{T}^*_{A^*-A}\,\mathscr{P} \\ 0 & \mathscr{P}\,\mathscr{T}^*_{A^*-A}\,\mathscr{P} & \mathscr{P}\,\mathscr{T}^*_{v-d\,(A^*-A)}\,\mathscr{P} \end{pmatrix}, \qquad \bar{g}^1_I := \begin{pmatrix} -\rho\,\nabla\cdot v_{II} \\ -\,(v\cdot\nabla)\cdot v_{II} \\ 0 \end{pmatrix}.$$

Recall that, given a smooth vector field C, the operator $\mathscr{T}_C$ is defined as in (3.3) with adjoint $\mathscr{T}^*_C = -C \times (\nabla\times\ )$. Consider the action of $\mathscr{P}\,\mathscr{T}^*_C\,\mathscr{P}$ where the presence of $\mathscr{P}$ eliminates the non symmetric one order terms (Remark 12). As a consequence, $\mathscr{P}\,\mathscr{T}^*_C\,\mathscr{P}$ is (modulo zero order terms) a skew-adjoint operator with principal symbol $i\,(C\cdot\xi)\ P(\xi)$. As required, A_I is skew-adjoint, the parabolic part on v_{II} is non-negative definite, while $\bar{g}_I$ depends only on $D_x v_{II}$. This is where the role of the bulk fluid viscosity can be understood. It is to compensate the losses of derivatives in the first equation. This idea can serve as a guide for obtaining the well-posedness.

There are however some specific issues among which the presence of the pseudo-differential action (4.15) to recover the coefficient A from A^*. For the sake of completeness, we give a direct proof in the next paragraph.

4.2.3. *L^2-energy estimates.* The energy $\mathcal{E}^{\mathfrak{c}}$ of (3.12) is a decreasing quantity. Assuming that ρ remains positive (recall that the internal energy U is positive as soon as $\rho > 0$), this provides with a priori estimates on v, ρ and $\nabla \times B$. Now, from (4.14) and (4.15), we can deduce that

$$\rho^{-1} \ \nabla \times B = (\mathscr{L}_2^{\mathfrak{c}})^{-1} \, \nabla \times (\nabla \times A^*) \, . \tag{4.22}$$

Then, by combining Proposition 14 and Lemma 15, we obtain as in the incompressible case some (high-frequency) L^2-bounds on v and A^*. But again, this is not sufficient. We would like to have extra controls on ρ (other than those furnished by the integral of $\rho\, U$) and especially stability estimates. For these reasons, we look at the linearized equations which are associated with (4.17). We think in terms of the unknowns (q, v, A^*), and therefore in terms of $\dot U_{\mathfrak{p}}^{\mathfrak{c}} := (\dot q, \dot v, \dot A^*)$. When doing this, this time, the term which is at top right of the second line of (4.17) leads to

$$\begin{aligned} -\,(A^* - A) \times (\nabla \times \dot A^*) + (\nabla \times A^*) \times \dot A^* \ & - (\nabla \times A^*) \times (\dot{\mathscr{L}}_2^{\mathfrak{c}})^{-1} \, (\rho \, A^*) \\ & - (\nabla \times A^*) \times (\mathscr{L}_2^{\mathfrak{c}})^{-1} \, (\dot\rho \, A^* + \rho \, \dot A^*) \, , \end{aligned}$$

where the dot on $\mathscr{L}$ is needed to keep track of the dependence of $(\mathscr{L}_2^{\mathfrak{c}})^{-1}$ on ρ. Given a Lipschitz field $U_{\mathfrak{p}}^{\mathfrak{c}}$, the three terms appearing in the right hand side are clearly bounded by the L^2-norm of $\dot U_{\mathfrak{p}}^{\mathfrak{c}}$, and they are therefore compatible with the local L^2-stability. To simplify the presentation, they are not mentioned. Modulo source terms (which are ignored), we can focus on

$$\left\{ \begin{aligned} & \partial_t \dot q + v \cdot \nabla \dot q + a(q) \ \nabla \cdot \dot v = 0 \, , \\ & \partial_t \dot v + (v \cdot \nabla) \dot v + a(q) \ \nabla \dot q - (A^* - A) \times (\nabla \times \dot A^*) \\ & \qquad\qquad\qquad + \nabla \big((A^* - A) \cdot (\dot A^* - \dot A) \big) = \nu \ \nabla (\nabla \cdot \dot v) \, , \\ & \partial_t \dot A^* - \big(v - d \, (A^* - A) \big) \times (\nabla \times \dot A^*) - (A^* - A) \times (\nabla \times \dot v) + \nabla \dot e = 0 \, . \end{aligned} \right. \tag{4.23}$$

At the initial time $t = 0$, we impose

$$\dot U_{\mathfrak{p}}^{\mathfrak{c}}(0, \cdot) = \dot U_{\mathfrak{p}0}^{\mathfrak{c}} = (\dot q_0, \dot v_0, \dot A_0^*) \in L^2(\mathbb{R}^3; \mathbb{R}) \times L^2(\mathbb{R}^3; \mathbb{R}^3)^2 \, . \tag{4.24}$$

Lemma 20. *[L^2-energy estimates for the linearized incompressible potential equations] Let $T > 0$. Assume that $U_{\mathfrak{p}}^{\mathfrak{c}} = (\rho, v, A^*)$ is such that $U_{\mathfrak{p}}^{\mathfrak{c}} \in C([0, T]; H^s)$ for some $s > 5/2$. Then, the Cauchy problem built with (4.15)-(4.23) and with initial data (4.24) has a solution on $[0, T]$. Moreover, we can find a constant C depending only on the $C([0, T]; \mathcal{H}^s)$-norm of $U_{\mathfrak{p}}^{\mathfrak{c}}$ such that*

$$\| \, \dot U_{\mathfrak{p}}^{\mathfrak{c}}(t, \cdot) \, \|_{L^2} \leq \| \, \dot U_{\mathfrak{p}0}^{\mathfrak{c}} \, \|_{L^2} \ e^{C\, t} \, , \qquad \forall \, t \in [0, T] \, . \tag{4.25}$$

Any $\mathcal{H}^s$-solution to the initial value problem (4.15)-(4.17)-(4.19) leads to a solution to (4.23)-(4.24) with initial data $\dot{U}^{\mathfrak{c}}_{\mathfrak{p}0} = U^{\mathfrak{c}}_{\mathfrak{p}0}$. As a consequence, the proof of Lemma 20 gives another access to some L^2-bound, namely

$$\| U^{\mathfrak{c}}_{\mathfrak{p}}(t,\cdot) \|_{L^2} \leq \| U^{\mathfrak{c}}_{\mathfrak{p}0} \|_{L^2} \ e^{C\,t} , \qquad \forall\, t \in [0,T] .$$

Proof. We multiply the first, second and third equation of (4.23) respectively by $\dot{q}$, $\dot{v}$ and $\dot{A}^*$; we integrate with respect to the variable x; and then we force everywhere the emergence of $\mathscr{T}_C$ by passing to the adjoint. This furnishes

$$\begin{aligned} \frac{1}{2}\ \frac{d}{dt}\Big(\int_{\mathbb{R}^3} |\dot{U}^{\mathfrak{c}}_{\mathfrak{p}}(t,\cdot)|^2 \ dx \Big) + \nu \int_{\mathbb{R}^3} |(\nabla\cdot\dot{v})(t,\cdot)|^2 \ dx \leq C \int_{\mathbb{R}^3} |\dot{U}^{\mathfrak{c}}_{\mathfrak{p}}(t,\cdot)|^2 \ dx \\ - \langle \mathscr{T}_{A^*-A} \dot{v}, \dot{A}^* \rangle - \langle \mathscr{T}_{v} \dot{A}^*, \dot{A}^* \rangle + \langle \mathscr{T}_{A^*-A} \dot{A}^*, d\,\dot{A}^* - \dot{v} \rangle . \end{aligned}$$

Knowing (4.18), the situation is exactly as in the incompressible case, except that the divergence of $\dot{v}$ is no more zero. The only new term which could be problematic is issued from the first contribution in the second line. It is unavoidable in our procedure. However

$$|\langle (\nabla\cdot\dot{v})\,(A^*-A), \dot{A}^* \rangle| \leq \frac{\nu}{2}\ \| \nabla\cdot\dot{v} \|^2_{L^2} + \frac{C}{\nu}\ \| \dot{U}^{\mathfrak{c}}_{\mathfrak{p}} \|^2_{L^2} ,$$

This is where the bulk (fluid) viscosity is indispensable. It serves to absorb the above loss of derivatives related to $\nabla\cdot\dot{v}$. By Grönwall's inequality, we recover (4.25). □

The comments in Paragraph 4.1.3 are still appropriate. Indeed, the compressible vorticity formulation is a derived version of the compressible potential formulation. As such, the work of Subsection 2.4 can serve to confirm that $\mathcal{H}^1$-estimates for the compressible potential formulation are available. This remark concludes the proof of Theorem 2.

5. Inertial wave phenomena

To better grasp the role of both d_i and d_e, in this section, we work with the spacetime variables of origin, those of (1.6). For $0 \leq d_e \lesssim d_i \ll 1$ and frequencies $|\xi| \ll d_i^{-1}$, XMHD like MHD involves principally Alvén and magnetosonic waves. The focus here is on what happens at higher frequencies, when $|\xi| \sim d_i^{-1}$ or $|\xi| \sim d_e^{-1}$, while usual MHD waves may be relegated to the back burner. The emphasis is on the emergence and propagation of inertial waves. To simplify, we address this issue in the incompressible context, with

$$\text{(5.1)} \qquad \begin{cases} \partial_t \mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v} + \nabla \mathrm{p} + \mathrm{B}^* \times (\nabla\times\mathrm{B}) = 0 , \\ \partial_t \mathrm{B}^* + \nabla\times\big(\mathrm{B}^*\times(\mathrm{v} - d_i\,\nabla\times\mathrm{B})\big) + d_e^2\ \nabla\times\big((\nabla\times\mathrm{v})\times(\nabla\times\mathrm{B})\big) = 0 , \end{cases}$$

together with (1.15) and

$$\text{(5.2)} \qquad \mathrm{B} = (\mathrm{Id} - d_e^2\,\Delta)^{-1}\mathrm{B}^* .$$

Our discussion is guided by the selection of different wave configurations, aimed at revealing various facets of the analysis. Each time, we follow the same guidelines. First, we exhibit particular solutions to (1.15)-(5.1)-(5.2). Secondly, we derive the corresponding linearized equations (this is an opportunity to come back and complete some aspects of the preceding analysis). Then, we study the inertial dispersion relations thus generated.

This strategy is implemented in different situations which become somewhat more and more sophisticated. We consider successively: constant solutions (Subsection 5.1), Beltrami fields (Subsection 5.2), configurations with null points (Subsection 5.3), a two dimensional framework (Subsection 5.4) and special moving solutions (Subsection 5.5).

5.1. **Constant solutions.** Of course, constant vector fields like $(\bar{\mathrm{v}}, \bar{\mathrm{B}}^*) \in \mathbb{R}^3 \times \mathbb{R}^3$ give rise to solutions. The associated linearized equations are readily identifiable

$$\left\{ \begin{array}{ll} \partial_t \dot{v} + (\bar{\mathrm{v}} \cdot \nabla) \dot{v} + \nabla \dot{p} + \bar{\mathrm{B}}^* \times (\nabla \times \dot{B}) = 0 \, , & \nabla \cdot \dot{v} = 0 \, , \\ \partial_t \dot{B}^* + (\bar{\mathrm{v}} \cdot \nabla) \dot{B}^* - (\bar{\mathrm{B}}^* \cdot \nabla)(\dot{v} - d_i \, \nabla \times \dot{B}) = 0 \, , & \nabla \cdot \dot{B}^* = 0 \, , \end{array} \right. \tag{5.3}$$

together with

$$\dot{B} = (\mathrm{Id} - d_e^2 \, \Delta)^{-1} \dot{B}^*. \tag{5.4}$$

The linear system (5.3) is not symmetric (and not directly symmetrizable), confirming that the unknowns $\dot{v}$ and $\dot{B}^*$ are not suitable. Following Subsection 2.3, we can introduce the weighted vorticity $\dot{w} := d_e \, \nabla \times \dot{v}$ to get

$$\left\{ \begin{array}{l} \partial_t \dot{w} + (\bar{\mathrm{v}} \cdot \nabla) \dot{w} = d_e^{-1} \, (\bar{\mathrm{B}}^* \cdot d_e \, \nabla) \big(d_e \, \nabla \times (\mathrm{Id} - d_e^2 \, \Delta)^{-1} \dot{B}^* \big) \, , \\ \partial_t \dot{B}^* + (\bar{\mathrm{v}} \cdot \nabla) \dot{B}^* = d_e^{-1} \, (\bar{\mathrm{B}}^* \cdot d_e \, \nabla) \big(\dot{v} - d \, d_e \, \nabla \times (\mathrm{Id} - d_e^2 \, \Delta)^{-1} \dot{B}^* \big) \, . \end{array} \right. \tag{5.5}$$

The derivatives of $\dot{v}$ (weighted by d_e) can be deduced from $\dot{w}$ as indicated in Lemma 23. Observe that the operators which are in factor of d_e^{-1} in the right hand side are uniformly (when $d_e \to 0$) bounded in L^2. Thus:

- For $\bar{\mathrm{B}}^* = O(d_e)$ or if the regime is weakly nonlinear as in (1.12), the source terms are uniformly bounded on any finite time interval. This is the framework of the present paper.
- For $\bar{\mathrm{B}}^* = O(1)$, the L^2-norm of $(\dot{w}, \dot{B}^*)$ may increase at a rate of d_e^{-1}. This depends on the structure (antisymmetric or not) of the source term. As already mentioned, the corresponding effects are connected to singularity formation [10, 23] or magnetic reconnection [16, 18]. These difficulties are not addressed here.

In other words, at very high frequencies $|\xi| \gtrsim d_e^{-1}$, due to the ellipticity induced by the constitutive relation, all standard hyperbolic contributions (managing usually Alfvén and magnetosonic waves) act in the right hand side as zero order terms. If $\bar{\mathrm{B}}^* = O(d_e)$, they remain under control. But, for $\bar{\mathrm{B}}^* = O(1)$, they could result (when $d_e \to 0$) in a very rapid amplification of the L^2-norm.

The linear system (5.5) is well-posed in L^2. However, its hyperbolic structure (the left hand side) is completely reduced, without any influence of d_e or d_i. We just find two decoupled transport equations at the velocity $\bar{v}$. The constant case is a point of entry that does not allow to catch rich phenomena. Still, it is illustrative of the role of source terms in the inertial regime.

5.2. **Beltrami fields.** Select some angular wave vector $\mathbf{k} \in \mathbb{R}^3$ whose angular wavenumber $k := |\mathbf{k}|$ is an integer ($k \in \mathbb{N}$), as well as some vector $Z_{\mathbf{k}} \in \mathbb{R}^3$ which is such that $\mathbf{k} \cdot Z_{\mathbf{k}} = 0$. With the help of $\mathbf{k}$ and $Z_{\mathbf{k}}$, we can construct the oscillatory wave

$$\mathscr{Z}_{\mathbf{k}}(x) := Z_{\mathbf{k}} \, \cos(\mathbf{k} \cdot x) + k^{-1} \, (Z_{\mathbf{k}} \times \mathbf{k}) \, \sin(\mathbf{k} \cdot x) \,.$$

This furnishes an eigenfunction of the curl operator with eigenvalue k, which is called a Beltrami field. From

$$\nabla \times \mathscr{Z}_{\mathbf{k}} = k \, \mathscr{Z}_{\mathbf{k}} \,, \qquad \nabla \cdot \mathscr{Z}_{\mathbf{k}} = 0 \,, \qquad 2 \, (\mathscr{Z}_{\mathbf{k}} \cdot \nabla) \mathscr{Z}_{\mathbf{k}} = \nabla |\mathscr{Z}_{\mathbf{k}}|^2 \,, \qquad \Delta \mathscr{Z}_{\mathbf{k}} = -k^2 \, \mathscr{Z}_{\mathbf{k}} \,,$$

we can deduce that $(\mathrm{v}, \mathrm{B}^*) = (\mathscr{Z}_{\mathbf{k}}, \mathscr{Z}_{\mathbf{k}})$ is a stationary solution to (1.15)-(5.1)-(5.2) with pressure $p = -|\mathscr{Z}_{\mathbf{k}}|^2/2$ and $B = (1 + d_e^2 \, k^2)^{-1} \, \mathscr{Z}_{\mathbf{k}}$. After some calculations, always with the weighted vorticity $\dot{w} := d_e \, \nabla \times \dot{v}$, we find that

$$\text{(5.6)} \qquad \begin{cases} \partial_t \dot{w} + (\mathscr{Z}_{\mathbf{k}} \cdot \nabla) \dot{w} + d_e \, k \, (\mathscr{Z}_{\mathbf{k}} \cdot \nabla) \dot{B}^* = \dot{\mathscr{S}}^{\mathrm{i}w}_{\mathfrak{v}0} \, (\dot{w}, \dot{B}^*) \,, \\ \partial_t \dot{B}^* + (1 - d_i \, k) \, (\mathscr{Z}_{\mathbf{k}} \cdot \nabla) \dot{B}^* + d_e \, k \, (\mathscr{Z}_{\mathbf{k}} \cdot \nabla) \dot{w} = \dot{\mathscr{S}}^{\mathrm{i}B^*}_{\mathfrak{v}0} \, (\dot{w}, \dot{B}^*) \,. \end{cases}$$

The operators $\dot{\mathscr{S}}^{\mathrm{i}\star}_{\mathfrak{v}0}$ are (as suggested by the notation) of order zero. They depend on d_e and d_i, and they show properties similar to those identified in Subsection 5.1. For $d_i = 0$, the two quantities $\dot{w} \pm \dot{B}^*$ satisfy two transport equations (coupled by source terms). These inertial waves travel along the same characteristics, those generated by $\mathscr{Z}_{\mathbf{k}}$, but with different speeds of propagation (due to the factor $1 \pm d_e \, k$ in front of $\mathscr{Z}_{\mathbf{k}} \cdot \nabla$).

5.3. **Null point configurations.** The locations where the magnetic field vanishes are called null points. Prototypes can (locally) take the form

$$\text{(5.7)} \qquad \mathrm{B}^{f*}_{\alpha} := {}^t(y, \alpha \, x, 0) \,, \qquad \mathrm{B}^{s*}_{\alpha} := {}^t(x, \alpha \, y, -(\alpha + 1) \, z) \,, \qquad \alpha \in \mathbb{R} \,.$$

The expression $(\mathrm{v}, \mathrm{B}^*)$ with $(\mathrm{v}, \mathrm{B}^*) = (0, \mathrm{B}^{f*}_{\alpha})$ or $(\mathrm{v}, \mathrm{B}^*) = (0, \mathrm{B}^{s*}_{\alpha})$ is a stationary solution satisfying $\mathrm{B} = \mathrm{B}^*$.

• <u>The case of</u> B^{f*}_{α}. First compute

$$\nabla \times \mathrm{B}^{f*}_{\alpha} = \begin{pmatrix} 0 \\ 0 \\ \alpha - 1 \end{pmatrix} , \qquad \mathrm{B}^{f*}_{\alpha} \times (\nabla \times \mathrm{B}^{f*}_{\alpha}) = (\alpha - 1) \begin{pmatrix} \alpha \, x \\ -y \\ 0 \end{pmatrix} = \frac{\alpha - 1}{2} \, \nabla (\alpha \, x^2 - y^2) \,.$$

It follows that

$$\text{(5.8)} \qquad \begin{cases} \partial_t \dot{w} + d_e \, (\alpha - 1) \, \partial_z \dot{B}^* = \dot{\mathscr{S}}^{\mathrm{i}w}_{\mathfrak{v}0} \, (\dot{w}, \dot{B}^*) \,, \\ \partial_t \dot{B}^* - d_i \, (\alpha - 1) \, \partial_z \dot{B}^* + d_e \, (\alpha - 1) \, \partial_z \dot{w} = \dot{\mathscr{S}}^{\mathrm{i}B^*}_{\mathfrak{v}0} \, (\dot{w}, \dot{B}^*) \,. \end{cases}$$

The inertial waves move (modulo possibly large source terms) in the vertical direction (the one of the stationary current density) at the speeds $d_i \pm (d_i^2 + 4 \, d_e^2)^{1/2} \, (\alpha - 1)/2$. In other words, two dimensional null points lend themselves to a transport of energy in the direction orthogonal to the (horizontal) magnetic surfaces. This effect disappears when the perturbation remains in the horizontal plane or in the particular case $\alpha = 1$ (when the separatrix angle is $\pi/2$).

• <u>The case</u> B^{s*}_{α}. This situation is even simpler since $\partial_t (\dot{w}, \dot{B}^*) = \dot{\mathscr{S}}^{\mathrm{i}}_{\mathfrak{v}0} \, (\dot{w}, \dot{B}^*)$.

Large amplitude magnetic fields like in (5.7) furnish usually templates in the perspective of reconnection models [40]. The problem is to describe what happens near the origin after perturbation. This would require (this is not done here) to measure the impact of the source term $\dot{\mathscr{S}}^{\mathfrak{i}}_{\mathfrak{v}}$ which is presumably of size d_e^{-1}.

5.4. **The two dimensional case.** We can also seek solutions which do not depend on z and which involve the following form (where B and B^* are both orthogonal to v)

$$\mathrm{v} = \begin{pmatrix} \mathrm{v}_1(t,x,y) \\ \mathrm{v}_2(t,x,y) \\ 0 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{b}(t,x,y) \end{pmatrix}, \qquad \mathrm{B}^* = \begin{pmatrix} 0 \\ 0 \\ \mathrm{b}^*(t,x,y) \end{pmatrix}. \tag{5.9}$$

Note that there exist two dimensional solutions of (5.1) which are more general than (5.9), by including the flux and stream functions (see [18]). With (5.9), the equations composing (2.11) reduce to the following 2×2 nonlinear system

$$\begin{cases} \partial_t \mathrm{w} + v_1\ \partial_x \mathrm{w} + v_2\ \partial_y \mathrm{w} + d_e\ (\partial_y \mathrm{b}\, \partial_x \mathrm{b}^* - \partial_x \mathrm{b}\, \partial_y \mathrm{b}^*) = 0\,, \\ \partial_t \mathrm{b}^* + (v_1 - d_i\, \partial_y \mathrm{b})\ \partial_x \mathrm{b}^* + (v_2 + d_i\, \partial_x \mathrm{b})\ \partial_y \mathrm{b}^* + d_e\ (\partial_y \mathrm{b}\, \partial_x \mathrm{w} - \partial_x \mathrm{b}\, \partial_y \mathrm{w}) = 0\,, \end{cases} \tag{5.10}$$

together with

$$\partial_x \mathrm{v}_1 + \partial_y \mathrm{v}_2 = 0\,, \qquad \mathrm{b} = (1 - d_e^2\, \Delta_{x,y})^{-1}\, \mathrm{b}^*\,. \tag{5.11}$$

For $d_e = 0$, we find that $b = b^*$, and the system (5.10) reduces to two transport equations

$$\begin{cases} \partial_t \mathrm{w} + v_1\ \partial_x \mathrm{w} + v_2\ \partial_y \mathrm{w} = 0\,, \\ \partial_t \mathrm{b}^* + v_1\ \partial_x \mathrm{b}^* + v_2\ \partial_y \mathrm{b}^* = 0\,. \end{cases} \tag{5.12}$$

The Hall effects (coming from d_i) just disappear (this is quite specific to this configuration). From now on, consider that $d_e > 0$. Then, the link between b and b^* is simplified making apparent the gain of two derivatives, and the role of $\partial_x \mathrm{b}$ and $\partial_y \mathrm{b}$ as coefficients. Moreover, we get simplifications since the source terms are eliminated. Fix five constants $(\bar{\mathrm{v}}_1, \bar{\mathrm{v}}_2, \alpha, \beta, \gamma) \in \mathbb{R}^5$ such that $-\bar{\mathrm{v}}_1 \gamma + \bar{\mathrm{v}}_2 \beta = 0$. The expressions

$$\bar{\mathrm{v}} = (\bar{\mathrm{v}}_1, \bar{\mathrm{v}}_2)\,, \qquad (\overline{\mathrm{w}}, \overline{\mathrm{b}}^*) := (0, \alpha - \gamma\, x + \beta\, y)\,, \qquad \overline{\mathrm{b}} = \overline{\mathrm{b}}^*\,, \tag{5.13}$$

give rise to solutions to (5.10) such that $\nabla \times \bar{\mathrm{B}}^* = {}^t(\beta, \gamma, 0)$ is constant. The linearized equations of (5.10) along these solutions are given by

$$\begin{cases} \partial_t \dot{w} + \bar{\mathrm{v}}_1\ \partial_x \dot{w} + \bar{\mathrm{v}}_2\ \partial_y \dot{w} + d_e\ (\beta\, \partial_x \dot{b}^* + \gamma\, \partial_y \dot{b}^*) - d_e\ (\beta\, \partial_x \dot{b} + \gamma\, \partial_y \dot{b}) = 0\,, \\ \partial_t \dot{b}^* + (\bar{\mathrm{v}}_1 - d_i\, \beta)\ \partial_x \dot{b}^* + (\bar{\mathrm{v}}_2 - d_i\, \gamma)\ \partial_y \dot{b}^* + d_e\ (\beta\, \partial_x \dot{w} + \gamma\, \partial_y \dot{w}) \\ \qquad + d_i\ (\beta\, \partial_x \dot{b} + \gamma\, \partial_y \dot{b}) = 0\,, \end{cases} \tag{5.14}$$

where $\dot{b} = (1 - d_e^2\, \Delta_{x,y})^{-1}\, \dot{b}^*$. In the quasilinear symmetric presentation (5.14), the two contributions $\beta\, \partial_x \dot{b}^* + \gamma\, \partial_y \dot{b}^*$ and $\beta\, \partial_x \dot{w} + \gamma\, \partial_y \dot{w}$ establish a balance, while $\partial_x \dot{b}$ and $\partial_y \dot{b}$ are viewed as source terms (of order zero). Given some angular wave vector $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2$ with angular wave number $k := |\mathbf{k}| \in \mathbb{R}_+$ and given $\tau \in \mathbb{C}$, we can seek plane wave solutions of the form

$$\dot{w} = \dot{w}_{\mathbf{k}}\ e^{i\, k_1\, x + i\, k_2\, y + i\, \tau\, t}, \qquad \dot{b}^* = \dot{b}^*_{\mathbf{k}}\ e^{i\, k_1\, x + i\, k_2\, y + i\, \tau\, t}\,. \tag{5.15}$$

Remark 21. *[Approximate vs complete dispersion relation] Neglecting the influence inside (5.14) of the zero order terms, we find the following two approximate dispersion relations*

$$\tilde{\tau}_\pm(\mathbf{k}) + \bar{\mathrm{v}} \cdot \mathbf{k} + \frac{1}{2}\, \kappa_\pm\, (\beta\, k_1 + \gamma\, k_2) = 0\,, \qquad \kappa_\pm := \frac{1}{2}\, \Big(d_i \pm \sqrt{d_i^2 + 4\, d_e^2} \Big)\,, \tag{5.16}$$

which are inherited from the symmetric form. As can be expected, the functions $\tilde{\tau}_\pm$ *are homogeneous of degree* 1 *with respect to* $\mathbf{k}$*.*

Now, observe that $\beta\, \partial_x \dot{b}^* + \gamma\, \partial_y \dot{b}^* - \beta\, \partial_x \dot{b} - \gamma\, \partial_y \dot{b} = -\, d_e^2\, \Delta_{x,y}\, (1 - d_e^2\, \Delta_{x,y})^{-1}\, (\beta\, \partial_x \dot{b}^* + \gamma\, \partial_y \dot{b}^*)$. Thus, after substitution of (5.15) inside (5.14), we get the condition $det\, (\tau\, \mathrm{Id}_{2\times 2} + A(\mathbf{k})) = 0$ where the matrix $A(\mathbf{k})$ is defined by

$$A(\mathbf{k}) := \bar{\mathrm{v}} \cdot \mathbf{k}\ \mathrm{Id}_{2\times 2} + (\beta\, k_1 + \gamma\, k_2)\, \begin{pmatrix} 0 & +d_e\, g(\mathbf{k}) \\ d_e & -d_i\, g(\mathbf{k}) \end{pmatrix}\,, \qquad g(\mathbf{k}) := \frac{d_e^2\ k^2}{1 + d_e^2\, k^2}\,.$$

We find two distinct <u>real</u> eigenvalues giving rise to the two complete dispersion relations

$$\tau_\pm(\mathbf{k}) + \bar{\mathrm{v}} \cdot \mathbf{k} + \frac{1}{2}\, (\beta\, k_1 + \gamma\, k_2)\, \Big[d_i\ g(\mathbf{k}) \pm \sqrt{d_i^2\, g(\mathbf{k})^2 + 4\, d_e^2\, g(\mathbf{k})} \Big] = 0\,. \tag{5.17}$$

This means that the addition of the zero order terms does not destroy the hyperbolic properties. The 2×2 system (5.14) is hyperbolic, with Fourier multipliers as coefficients:

- For $\mathbf{k}$ orthogonal to $\nabla \times \bar{\mathrm{B}}^*$, we just find $\tilde{\tau}_\pm(\mathbf{k}) = \tau_\pm(\mathbf{k}) = -\bar{\mathrm{v}} \cdot \mathbf{k}$.
- For $d_e > 0$, we must incorporate supplementary corrections on both $\tilde{\tau}_\pm$ and $\tau_\pm$ that characterize the propagation of inertial waves. Moreover, at the level of $\tau_\pm$, we observe dispersive effects encoded in the (non constant) behavior of g. On the other hand, for $k \gg d_e^{-1}$, we get $g(\mathbf{k}) \sim 1$. As a consequence, the asymptotic description of $\tau_\pm(\mathbf{k})$ gives way to $\tilde{\tau}_\pm(\mathbf{k})$.

Remark 22. *[On the determination of the complete dispersion relations] Keep in mind that extra dispersive effects may be induced by the source terms which have been skipped in this subsection. This is here illustrated by the difference between* $\tilde{\tau}_\pm$ *and* $\tau_\pm$*. The eigenvalues* $\lambda_\pm$ *of (2.19) only provide information on the (maximal possible) homogeneous behavior (of order* 1*) inherited by the speeds of propagation (for conveniently polarized waves).*

5.5. **Moving solutions.** Consider that $d_i = 1$ and $d_e \ll 1$. Let B_0 be a fixed constant magnetic field. In [4, 3], given $\mathbf{k} \in \mathbb{R}^3$, the authors seek plane wave solutions having the following form (where, on condition that $\mathrm{B}_0 = 0$, B and v are parallel)

$$\mathrm{B} = \mathrm{B}_0 + \mu_{\mathbf{k}}^\pm\ v_{\mathbf{k}}^\pm\ e^{i\, \mathbf{k}\cdot x + i\, \mu_{\mathbf{k}}^\pm\, (\mathrm{B}_0 \cdot \mathbf{k})\, t}\,, \qquad \mathrm{v} = \mathrm{v}_{\mathbf{k}}^\pm\ e^{i\, \mathbf{k}\cdot x + i\, \mu_{\mathbf{k}}^\pm\, (\mathrm{B}_0 \cdot \mathbf{k})\, t}\,, \qquad \mathrm{v}_{\mathbf{k}}^\pm \in \mathbb{R}^3. \tag{5.18}$$

By adjusting the value of $\mu_{\mathbf{k}}^\pm$ adequately, they show that such solutions do exist. Observe that $\mathrm{B} = \mathrm{B}_0 + \mu_{\mathbf{k}}^\pm\, \mathrm{v}$, so that $(\nabla \times \mathrm{B}) \times (\nabla \times \mathrm{v}) = 0$. This means that the choice (5.18) has the effect of killing some nonlinearities and in fact, remarkably, all nonlinearities. The choice (5.18) is to some extent the opposite of (5.9). This polarization eliminates the terms which are emphasized at the level of (2.11) or (5.10), those with d_e in factor. The dynamics induced by (5.18) have nothing to do with inertial waves. Rather, they are tied to some extension of Alfvén waves.

For $\mathbf{k} = k\, e_z$ with $e_z = {}^t(0,0,1)$ and where $k = |\mathbf{k}| \in \mathbb{R}$ stands again for the angular wavenumber, we get a special type of waves with associated dispersion relation

$$\omega^{\pm}_{\mathbf{k}} = -\mu^{\pm}_{\mathbf{k}}\, (\mathrm{B}_0 \cdot \mathbf{k}) = \frac{-k}{1 + d_e^2\, k^2} \; \left[-\frac{k}{2} \pm \sqrt{\frac{k^2}{4} + (1 + d_e^2\, k^2)} \; \right] \; (\mathrm{B}_0 \cdot e_z)\,, \tag{5.19}$$

which clearly exhibits dispersive properties. In Section 3.2 of [3], some comments are given about (5.19), which corresponds to a generalization of the dispersion relation for shear Alfvén waves in ideal MHD. Since the $\omega^{\pm}_{\mathbf{k}}$ remain bounded, the role of electron inertia in this case is to impose a lower and upper bound on the time frequencies attainable. In contrast, in the Hall framework, we get $\omega^{-}_{\mathbf{k}} = k^2\, (B_0 \cdot e_z)$ which rapidly diverges as the spatial wavenumber k tends to infinity. This means that the electron inertia has the effect on $\mu^{-}_{\mathbf{k}}$ to cure singular behaviors at high wave numbers in Hall MHD.

6. Appendix

In Subsection 6.1, we list some useful identities implying $\nabla\times$. In Subsection 6.2, we recall elliptic L^2-estimates concerning the div-curl system. These estimates have been exploited to control the derivatives of v in terms of $\nabla \times v$ and $\nabla \cdot v$.

6.1. **Identities involving the curl operator.** Retain that

$$\nabla \times (\nabla \times v) = \nabla(\nabla \cdot v) - \Delta v\,, \tag{6.1}$$

We need to know that

$$\nabla \times (F \times G) = \big((\nabla \cdot G) + G \cdot \nabla\big) F - \big((\nabla \cdot F) + F \cdot \nabla\big) G\,. \tag{6.2}$$

Recall also that

$$\begin{aligned} \nabla \times (F \cdot \nabla G) = \; & (F \cdot \nabla) \nabla \times G - ((\nabla \times G) \cdot \nabla) F \\ & + (\nabla \times G)(\nabla \cdot F) + \sum_{i=1}^{3} \nabla F_i \times \nabla G_i\,. \end{aligned} \tag{6.3}$$

6.2. **Elliptic estimates for the div-curl system.** Consider in $\mathbb{R}^3$ the system

$$\left\{ \begin{array}{l} \nabla \times v = w\,, \\ \nabla \cdot v = g\,, \end{array} \right. \tag{6.4}$$

where w and g are given data in L^2, whereas v is the unknown. From (6.1), we get that

$$\sum_{i,j}^{3} \int_{\mathbb{R}^3} |\partial_i v_j|^2 \; dx = \int_{\mathbb{R}^3} \big(|\nabla \cdot v|^2 + |\nabla \times v|^2\big) \; dx = \int_{\mathbb{R}^3} \big(|w|^2 + g^2\big) \; dx\,.$$

The derivatives of v in L^2 are therefore controlled by the L^2-norms of w and g. Using the Poincaré–Sobolev inequality (i.e. $\|v\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla v\|_{L^2(\mathbb{R}^3)}$) we also obtain the control of the L^2-norm of v; hence its $\mathcal{H}^1$-norm. Below, we formalize this well-known fact [43]. For the sake of completeness, we also give a more explicit proof of it.

6.2.1. *Link between the vorticity and the derivatives of a divergence free velocity.* We start by manipulating solenoidal vector fields, belonging to $\mathcal{D}^s$. The link between w and v is then achieved through the Biot-Savart law.

$$v = \nabla \times (-\Delta)^{-1} w \,. \tag{6.5}$$

Lemma 23. *[Continuity properties when passing from $\nabla \times v$ to $\partial_i v$] Given $w \in \mathcal{D}^s$, there exists a unique solenoidal vector field v such that $w = \nabla \times v$ in the distributional sense. Moreover, for all $i \in \{1,2,3\}$, the linear operator $\mathcal{M}_i^{\mathfrak{i}} : \mathcal{D}^s \to \mathcal{D}^s$ which sends w to $\partial_i v$ (with v as above) may be defined as a bounded matrix Fourier multiplier. It is therefore continuous for all $s \in \mathbb{R}$.*

Proof. Fix any $w \in \mathcal{D}^s$. By Poincaré lemma, we can find some v such that $w = \nabla \times v$ in the distributional sense. If we impose moreover $\nabla \cdot v = 0$, on the Fourier side, we have to deal with the explicit relation $\hat{v} = \mathcal{F}v = i\,|\xi|^{-2}\,\mathcal{F}w \times \xi$, which furnishes

$$\widehat{\partial_i v} = M_i^{\mathfrak{i}}(\xi)\,\mathcal{F}w\,, \qquad M_i^{\mathfrak{i}}(\xi) := -\frac{\xi_i}{|\xi|^2}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

It is clear that the matrix-valued function $M_i^{\mathfrak{i}}$ is bounded on $\mathbb{R}^3 \setminus \{0\}$. □

6.2.2. *Link between (w, g) and the derivatives of v.* The compressible version of Lemma 23 is the following.

Lemma 24. *[Continuity properties when passing from the couple $(\nabla \times v, \nabla \cdot v)$ to $\partial_i v$] Let $g \in \mathcal{H}^s(\mathbb{R}^3;\mathbb{R})$ and $w \in \mathcal{D}^s$. There exists a unique v such that $(\nabla \times v, \nabla \cdot v) = (g, w)$ in the distributional sense. Moreover, for all $i \in \{1,2,3\}$, the linear operator $\mathcal{M}_i^{\mathfrak{c}} : \mathcal{H}^s \times \mathcal{H}^s \to \mathcal{H}^s$ which sends (g, w) to $\partial_i v$ (with v as above) may be defined as a bounded matrix Fourier multiplier. It is therefore continuous.*

Proof. By construction, we have $\mathcal{M}_i^{\mathfrak{c}}(g, w) = \mathcal{F}^{-1}\big(-\xi_i\,(\hat{g}\,\xi - \xi \times \hat{w})/|\xi|^2\big)$ which is sufficient to conclude. □

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(Nicolas Besse) Observatoire de la Côte d'Azur, Bd de l'Observatoire CS 34229, 06304 Nice Cedex 4, France

(Christophe Cheverry) Institut Mathématique de Rennes, Campus de Beaulieu, 263 avenue du Général Leclerc CS 74205 35042 Rennes Cedex, FRANCE